Asset Pricing with Dynamic Margin Constraints

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ABSTRACT

This paper provides a novel theoretical analysis of how endogenous time-varying margin requirements affect capital market equilibrium. I find that margin requirements, when there are no other market friction, reduce the volatility and correlation of returns as well as the risk-free rate, but increase the market price of risk, the risk premium, and the price of risky assets. Furthermore, margin requirements generate a strong cross-sectional dispersion of stock return volatilities. The results emphasize that a general equilibrium analysis may reverse the conclusions of a partial equilibrium analysis often employed in the literature.

Margin debt is widely used in the financial industry to build leveraged positions. To protect themselves against losses caused by adverse price movements, creditors impose margin requirements, which put an upper limit on borrowers’ portfolio leverage. The tightness of margin constraints may be regulated legislatively or determined by an agreement between the borrower and the lender. For instance, Regulation T of the Federal Reserve sets initial margin requirements on equity investments, that is, it restricts the loan amount that investors can receive from brokers for purchasing stocks. Since January 1974, the initial margin requirement has been 50%. In addition, brokerage firms usually impose further restrictions on how leveraged their customers can be by setting a minimum amount of equity that borrowers must maintain.

The tightness of margin requirements is often influenced by current market conditions and may change over time. Consider, for example, hedge funds that borrow from prime brokerage firms and structure the debt as repurchase agreements (repos). Effectively, repos represent a form of collateralized borrowing: a borrower sells a security to a lender with a commitment to buy it back at a later date for a prespecified price. The market value of the security typically

*Oleg Rytchkov is at the Fox School of Business, Temple University. This paper draws upon my earlier paper circulated under the title “Dynamic Margin Constraints.” I am grateful to Nina Baranchuk, Lorenzo Garlappi, Ilan Guedj, Jennifer Huang, Leonid Kogan, Igor Makarov, Jun Pan, Dimitris Papanikolaou, Steve Ross, Astrid Schornick (discussant), Elizaveta Shevyakhova, Sophie Shive (discussant), Sheridan Titman, Stathis Tompaidis, Jiang Wang, seminar participants at Boston University, New Economic School, Nova Southeastern University, Purdue University, Temple University, University of Texas at Austin, and University of Texas at Dallas, as well as participants of the 2009 European Finance Association Meetings and 2009 Financial Management Association Meetings for very helpful and insightful comments. I especially thank Campbell Harvey (the Editor), an anonymous Associate Editor, and two anonymous referees for many comments and suggestions that substantially improved the paper. All remaining errors are my own.

DOI: 10.1111/jofi.12100

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exceeds the amount of cash that the seller receives; the difference is referred to as a haircut. Thus, repos are overcollateralized and the haircut is an analog of margin requirements, as it effectively imposes limits on permissible leverage. Repos are short-term transactions, and their terms are determined by continuous bilateral negotiations between hedge funds and prime brokerage firms. Prime brokers are generally permitted to modify the haircut (change margin requirements) at their own discretion and even without advance notice, and they are more likely to use this right when market conditions change.\footnote{Gorton and Metrick (2010) demonstrate how haircuts on various asset classes changed during the recent financial crisis.}

Although time variation in margin requirements is quite common, its equilibrium asset pricing implications have received little attention in the academic literature. This paper aims to fill this gap. My main objective is to study theoretically the general equilibrium effects of margin requirements that are time varying and endogenously determined by market conditions, which in turn are determined by the optimal behavior of investors, who take into account the fact that margins are influenced by market conditions. To conduct the analysis, I use a dynamic model of a pure exchange economy with one Lucas tree and heterogeneous investors who have constant relative risk aversion (CRRA) preferences with different levels of risk aversion. Due to the heterogeneity in preferences, investors trade with each other to share risks. In the equilibrium, less risk-averse agents borrow from those who are more risk-averse but may face margin constraints, which represent the only market imperfection in the model. The level of margin requirements is linked to endogenously determined current market conditions (e.g., the volatility of returns). Since market conditions vary across states of the economy, margin constraints are also state dependent. It should be emphasized that the results are quite general as I do not specify the exact functional form of margin requirements in the main analysis.

The paper contains several new results. First, I find that binding margin requirements decrease the volatility of returns. This is a very general effect: it holds for a wide class of margin constraints whose levels are linked to current market conditions. The intuition behind this effect is as follows. Due to the heterogeneity in investor preferences, less risk-averse investors borrow from those who are more risk-averse and invest in the risky asset. The optimal portfolios of the two groups of agents may be very different, and the dynamic risk sharing between them makes returns more volatile than fundamentals (e.g., Bhamra and Uppal (2009, 2010), Longstaff and Wang (2012)). Binding margin constraints reduce the difference in the portfolios that the two groups hold in the equilibrium, thereby curbing the excessive volatility produced by risk sharing.

This result has an important implication. In particular, it suggests that, in contrast to widespread beliefs, binding margin constraints, even when state dependent, cannot in and of themselves increase the volatility of returns. This does not mean that in practice margins never induce excessive volatility, but rather that they do so only indirectly through their interactions with other
market imperfections. For example, the volatility of returns may increase due to the “disutility of trading” (Aiyagari and Gertler (1999)), asymmetric information (Brunnermeier and Pedersen (2009)), or a stochastic proportion of less risk-averse investors in the economy (Kupiec and Sharpe (1991)). Thus, the results suggest that, instead of asking whether margin requirements increase or decrease volatility, future research should focus on identifying realistic market imperfections that may increase volatility of returns through their interaction with margin constraints.

Along with volatility, margin constraints affect other equilibrium variables: the risk-free rate is lower in the economy with margins than in the unconstrained economy, whereas the market price of risk and expected excess returns are higher. These effects also admit intuitive economic explanations. In particular, the increase in expected excess returns and the market price of risk can be attributed to the redistribution of the risky asset from less risk-averse investors to those who are more risk-averse and require a higher risk premium. The decrease in the risk-free rate is caused by lower demand for credit: margin constraints limit the leverage of less risk-averse investors, so they borrow less, and this drives the risk-free rate downward.

The opposite effects of margin constraints on the risk-free rate and expected excess returns have an important implication. Several studies (e.g., Kupiec and Sharpe (1991), Chowdhry and Nanda (1998), Wang (2013)) argue that margin requirements may increase stock return volatility when the interest rate is assumed to be constant and fixed. Effectively, the higher volatility of prices is produced by more volatile expected excess returns. However, when an increase in expected excess returns is offset by a decrease in the risk-free rate, the volatility of total expected returns may decrease, and this may make the price of the risky asset less volatile. Thus, my results emphasize the importance of using a general equilibrium framework to analyze margin requirements and suggest that conclusions from a partial equilibrium analysis may be misleading.

Next, I explore the impact of constrained investors’ hedging demand on the equilibrium properties. I compare the same economy with its analog, in which less risk-averse investors are replaced by myopic investors, who cannot anticipate changes in investment opportunities. I demonstrate that without margin constraints the hedging demand does not affect the equilibrium. However, hedging of time-varying investment opportunities by constrained investors does change the equilibrium and amplifies the impact of constraints. This result justifies the use of general CRRA preferences instead of logarithmic preferences, which are widely employed in the literature on economies with constraints.

To study the cross-sectional effects of margin requirements, I consider an extension of the model with two risky assets. I assume that their fundamentals move independently and that margin requirements limit both long and short positions. This analysis delivers several new insights. First, margin requirements not only reduce the volatility of returns on both assets, but also produce dispersion of volatility in the cross-section. When both long and short
positions tie investors' capital, it is optimal for less risk-averse investors to form a leveraged portfolio by investing almost all their wealth in a high-beta stock (a stock with a large dividend share in the model) and ignore the benefits of diversification. As a result, investor heterogeneity and risk shifting amplify the volatility of only one asset, that with the highest beta, whereas the volatility of the other asset can be even lower than the volatility of its fundamentals.

Second, margin requirements reduce the correlation between asset returns, and this effect is particularly strong when the assets have comparable size. Even though the asset fundamentals are uncorrelated, their returns are positively correlated because (i) investors hold diversified portfolios and simultaneously rebalance positions in both stocks (e.g., Cochrane, Longstaff, and Santa-Clara (2008)), and (ii) investors are heterogeneous and dynamically share risks by buying and selling both stocks (e.g., Ehling and Heyerdahl-Larsen (2012)). Because of margin requirements, less risk-averse investors trade mostly the high-beta stock, so both effects are suppressed and the correlation between returns decreases.

This paper is related to a vast literature that explores the structure of equilibrium in pure exchange economies with rational investors who have heterogeneous preferences and/or heterogeneous beliefs.\(^2\) It also extends the literature that studies the impact of constant portfolio constraints in a general equilibrium framework.\(^3\) The majority of existing papers assume that constrained investors have logarithmic preferences. Although this assumption often delivers tractability, it limits the scope of analysis, making it impossible to explore the impact of hedging motives on equilibrium properties. An exception is Chabakauri (2013), who extends the analysis to models with constrained investors and arbitrary levels of risk aversion.

One of the main contributions of this paper is the analysis of time-varying endogenous margin constraints. Although such constraints are common in financial markets, they are surprisingly underexplored in the academic literature. Basak and Shapiro (2001) study the optimal portfolio policy and equilibrium implications of restrictions on value-at-risk over a finite horizon, which can be interpreted as a form of dynamic portfolio constraints. Danielsson, Shin, and Zigrand (2004) and Danielsson and Zigrand (2008) examine the impact of a value-at-risk constraint imposed over a one-period horizon. Danielsson, Shin, and Zigrand (2011), Kyle and Xiong (2001), and Xiong (2001) examine models in which one type of investor is characterized by an aggregate demand curve.


Shin, and Zigrand (2011) and Prieto (2011) explore the equilibrium effect of constraints in the form of restrictions on the volatility of portfolio returns. Yuan (2005) constructs a one-period rational expectation model in which the borrowing constraint is a function of price.

This paper also contributes to the literature on the relation between margin constraints and the volatility of stock returns. Using models with various special assumptions and combinations of market imperfections, many existing studies claim that margin constraints can amplify volatility. My results highlight that seemingly innocuous model ingredients, like a constant exogenous risk-free rate, are essential for this conclusion to hold. For example, using an overlapping generations framework with heterogeneous agents, Kupiec and Sharpe (1991) show that the relation between constant margin requirements and stock price volatility is ambiguous and depends on the source of variation in the aggregate risk-bearing capacity. In particular, if the proportion of more risk-averse investors relative to the population changes exogenously and stochastically over time, then the imposition of constant margin requirements can increase the volatility of returns. However, to arrive at this conclusion the authors need to assume that the risk-free rate is exogenously fixed. Chowdhry and Nanda (1998) show that, in a two-period model, constant margin requirements can produce market instability, which is understood as a multiplicity of market-clearing prices. However, one of their important assumptions is the existence of a storage technology, which is equivalent to the exogeneity of the risk-free rate. Wang (2013) develops a one-period model with “liquidity suppliers” and “liquidity demanders,” who trade to hedge a payoff of an additional non-traded asset. In this setting, margin requirements increase volatility when they are imposed on liquidity suppliers whose population weight is small. Similar to the other papers, Wang (2013) assumes that the risk-free rate is exogenous and constant. Aiyagari and Gertler (1999) introduce “disutility of trading” in the preferences of one agent (representative household), which is tantamount to transaction costs and limits the speed at which the household can respond to a deviation of the market price from the fundamental price. Disutility of trading hampers the ability of the household to absorb shocks to prices when the other agent (trader) hits his margin constraint and is forced to reduce his security holdings. As a result, asset prices overreact and volatility increases.

Brunnermeier and Pedersen (2009) build a four-period model in which the tightness of margin constraints is endogenously determined by financiers, who try to limit their counterparty credit risk. The key assumption of the model producing destabilizing margins is information asymmetry between leveraged traders (speculators) and financiers: when the latter are uninformed, they cannot distinguish fundamental shocks from liquidity shocks and hence tighten constraints rather than provide liquidity, and this produces a destabilizing effect on asset prices. As in the previous papers, the interest rate is exogenous and fixed.

There is a long-standing debate in the empirical literature about the influence of margin requirements mandated by the Federal Reserve on stock market volatility. Many authors claim that there is no robust evidence in favor
of margin requirements as an effective policy instrument (e.g., Moore (1966), Schwert (1989), Hsieh and Miller (1990)). This view is challenged by Hardouvelis (1990), who argues that historically more stringent margin requirements are associated with lower stock market volatility. This conclusion is partially supported by Hardouvelis and Theo
dossiou (2002), who demonstrate that there is a negative (no) causal relation between tightness of margins and volatility in bull (bear) markets. Similarly, using data from the Tokyo Stock Exchange, Hardouvelis and Peristiani (1992) show that an increase in margin requirements is followed by a decline in the volatility of daily returns. My theoretical results can provide an explanation for these findings.

The rest of the paper is organized as follows. Section I presents the main model with endogenously determined margin constraints, describes the structure of the equilibrium, and provides analytical results. Section II reports numerical results. Section III presents an extension of the model with two risky assets and analyzes the cross-sectional effects of margin requirements. Section IV concludes by summarizing the main insights of the paper. Appendix A collects all proofs and Appendix B provides details on numerical techniques.

I. Model

In this section, I present the main model of the paper. I consider a continuous-time pure exchange economy with an infinite horizon and heterogeneous investors. The key component of the model is state-dependent margin constraints, whose level is endogenously determined by equilibrium conditions.

A. Assets

There are two assets in the economy. The first is a short-term risk-free asset (bond) in zero net supply with a rate of return $r_t$ determined in the equilibrium. The second asset (stock) is risky, and the aggregate supply of this asset is normalized to one. The risky asset produces a consumption good paid as a dividend $D_t$. The flow of the dividend follows a geometric Brownian motion

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_D dB_t,$$

where $B_t$ is a standard Brownian motion defined on the probability space $(\Omega, F, P)$. The drift $\mu_D$ and the volatility $\sigma_D$ are constant. In general, the price of the risky asset $S_t$ is a function of the current dividend $D_t$ and state variables. It is convenient to introduce a cum dividend excess return on one dollar invested in the risky asset:

$$dQ_t = \frac{dS_t + D_t dt}{S_t} - r_t dt.$$

The stochastic differential equation for $Q_t$ can be written as

$$dQ_t = \mu_Q dt + \sigma_Q dB_t,$$
where $\mu_Q$ is the risk premium and $\sigma_Q$ is the instantaneous volatility of returns. Both $\mu_Q$ and $\sigma_Q$ are functions of state variables and determined by equilibrium conditions.

**B. Agents**

There are two types of agents in the model, with the agents of one type (type A) more risk-averse than the agents of the other type (type B) (following Garleanu and Pedersen (2011), they can be dubbed Averse and Brave, respectively). There is an infinite number of identical investors of both types, and they all behave competitively. Each moment $t$, investor $i$ allocates his wealth between stocks and bonds with portfolio weights $\omega_{it}$ and $1 - \omega_{it}$, respectively. The optimal portfolio policies and consumption streams maximize investors’ utility functions. Both types of investors have standard CRRA preferences over intermediate consumption

$$U_{it} = E_t \left[ \int_t^\infty e^{-\beta s} u(C_{is}) \, ds \right], \quad u(C) = \frac{C^{1-\gamma} - 1}{1-\gamma},$$

but differ in terms of the coefficient of risk aversion $\gamma$: $\gamma = \gamma_A$ for type A investors and $\gamma = \gamma_B$ for type B investors, where $\gamma_A > \gamma_B$.\(^4\) All investors have the same time preference parameter $\beta$. Given the dividend process from equation (1), the expected utility is uniformly bounded if

$$\beta > \max \left(0, (1 - \gamma) \left( \mu_D - \frac{1}{2} \gamma \sigma_D^2 \right) \right),$$

and this condition should be satisfied for both types of investors.

Heterogeneity in investor preferences plays a crucial role in the model. Type B investors can be thought of as hedge funds or sophisticated proprietary traders, who are professional risk takers that aggressively invest in the risky asset. When type B investors need to sell stocks (e.g., due to binding margin constraints), type A investors take the other side of the trade. Thus, the latter are buyers of last resort and can be identified as pension funds, sovereign funds, or large individual investors. Trading and risk sharing between investors with different risk attitude are responsible for nontrivial dynamics of the model and make margin requirements relevant to investors.

**C. State Variable**

Following the literature on asset pricing with heterogeneous agents, I introduce a relative consumption share of one type of investor as a state variable.

\(^4\) In the Internet Appendix, I consider an extension of the model with hyperbolic absolute risk aversion (HARA) preferences and show that my main results are not specific to the CRRA utility function. The Internet Appendix is available in the online version of the article on *The Journal of Finance* website.
Specifically, I define $s_t$ as type B investors’ share of aggregate consumption:

$$s_t = \frac{C_t^B}{C_t^A + C_t^B} = \frac{C_t^B}{D_t},$$

where $C_t^A$ and $C_t^B$ are aggregate equilibrium consumption streams of type A and type B investors, respectively. Later, I will show that this is the only variable needed to describe the state of the economy. In particular, the interest rate $r_t$ is a function of $s_t$: $r_t = r(s_t)$. Similarly, the expected excess returns on the risky asset $\mu_Q(s_t)$ and the volatility of returns $\sigma_Q(s_t)$ are functions of $s_t$ only, and these functions are determined in the equilibrium. In general, the dynamics of the state variable $s_t$ can be represented as

$$ds_t = \mu_s(s_t)dt + \sigma_s(s_t)dB_t,$$

where the functions $\mu_s(s_t)$ and $\sigma_s(s_t)$ are also determined by equilibrium conditions.

### D. Margin Constraints

The key feature of the model is the presence of state-dependent margin requirements. When investors find the risky asset attractive, they cannot only reduce current consumption to buy stocks, but also issue bonds (borrow consumption good). Because there are only two types of investors in the model and bonds are in zero net supply, investors of one type are lenders, whereas investors of the other type are borrowers. At each moment, lenders put restrictions on how leveraged borrowers can be, that is, they impose margin requirements. Neither the financing process nor the contracting between borrowers and lenders is specified in the model. Instead, I assume that at any moment $t$ and for any investor $i$, the weight of the risky asset $\omega_{it}$ in the investor’s portfolio cannot exceed a certain threshold:

$$\omega_{it} \leq \bar{\omega}(\cdot, \sigma_Q(s_t), \ldots).$$

The function $\bar{\omega}(\cdot)$ determines the form of margin requirements and exogenously links their tightness to various variables, such as the volatility of returns, the volatility of the state variable, etc.\(^5\) Because the arguments of the function $\bar{\omega}(\cdot)$ are endogenous and determined by equilibrium conditions, the level of margin requirements is state dependent and endogenous. To ensure

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\(^{5}\) A limit on the weight of the risky asset is equivalent to a limit on the loan-to-value ratio. Indeed, equation (5) implies that $N_{it}S_t/W_{it} \leq \bar{\omega}(\cdot)$, where $W_{it}$ is the investor’s wealth, which can be represented as $W_{it} = N_{it}S_t - B_{it}$, and $B_{it}$ is the value of debt used to partially finance the purchase of $N_{it}$ shares of the risky asset. Hence, equation (5) can be rewritten as $B_{it}/(N_{it}S_t) \leq 1 - 1/\bar{\omega}(\cdot)$, that is, it sets an upper limit on the loan-to-value ratio.

\(^{6}\) In the Internet Appendix, I examine a version of the model with random jumps in margin requirements and show that the assumption of a deterministic link between constraints and equilibrium variables is not crucial to obtain the main results of the paper.
that the constraints put restrictions on leverage but not on market participation, I assume that \( \tilde{\omega}(\cdot) > 1 \) for all \( s \in [0, 1] \).

### E. Optimal Consumption and Portfolio Problem

The optimal consumption and portfolio problem of investors is standard, except that some investors may be constrained by margin requirements. Each investor maximizes CRRA utility function (3) subject to the state-dependent margin constraint (5) and the standard budget constraint

\[
dW_{it} = (r(s_t)W_{it} - C_{it})dt + \omega_{it}W_{it}(\mu_Q(s_t)dt + \sigma_Q(s_t)dB_t),
\]

where \( W_{it} \) is the total wealth of investor \( i \) and \( \omega_{it} \) is the share of total wealth invested in the risky asset. Note that for each investor the margin constraint is completely characterized by the function \( \tilde{\omega}(s) = \tilde{\omega}(\sigma_s(s), \sigma_Q(s), \ldots) \). The solution to the portfolio problem is given by the following proposition, where time subscripts and individual investor indexes are omitted to simplify notation.

**Proposition 1:** The solution to the consumption and portfolio problem of an investor maximizing the CRRA utility function (3) subject to the budget constraint (6) and the margin requirement (5) is

\[
C = \exp \left( -\frac{H(s)}{\gamma} \right) W,
\]

\[
\omega(s) = \min \left( \tilde{\omega}(s), \omega^*(s) \right), \quad \omega^*(s) = \frac{\mu_Q(s)}{\gamma\sigma_Q(s)^2} + \frac{\sigma_s(s)}{\gamma \sigma_Q(s)} H'(s),
\]

where a twice continuously differentiable function \( H(s) \) solves the ordinary differential equation

\[
\frac{1}{2} \sigma_s(s)^2 \left( H''(s) + H'(s)^2 \right) + H'(s) \mu_s(s) + r(s)(1 - \gamma) - \beta + \gamma \exp \left( -\frac{H(s)}{\gamma} \right) = 0,
\]

and the indirect utility function of the investor is

\[
J(s, W, t) = \frac{1}{1 - \gamma} W^{1-\gamma} \exp(H(s)) \exp(-\beta t).
\]

**Proof:** See Appendix A

Proposition 1 shows that the solution to the portfolio problem includes an endogenously determined boundary: in order to choose the optimal portfolio policy, the investor needs to decide not only how much to invest in the risky asset when the constraint does not bind, but also in which states \( s \) it is optimal to
allow the constraint to bind. Both choices are encoded in the function $H(s)$. The point at which the portfolio hits the constraint is determined by the equation $\bar{\omega}(s) = \omega^*(s)$. Thus, the optimal portfolio policy in equation (8) has a two-region structure. When the margin constraint is sufficiently tight ($\bar{\omega}(s)$ is sufficiently low), the investor prefers to invest up to this limit and $\omega(s) = \bar{\omega}(s)$. However, when the margin constraint is loose, the optimal portfolio is $\omega(s) = \omega^*(s)$, where $\omega^*(s)$ is given by equation (8).

The unconstrained demand $\omega^*(s)$ has an intuitive structure. The first term is a standard myopic demand. The second term is a hedging demand produced by time variation in (i) the interest rate $r(s)$, (ii) the market price of risk $\mu_Q(s)/\sigma_Q(s)$, and (iii) the portfolio constraint $\bar{\omega}(s)$. The first two hedging motives are standard and have been extensively analyzed in the literature (e.g., Merton (1971), Detemple, Garcia, and Rindisbacher (2003), Liu (2007)). The third motive has received much less attention. Since the interest rate, market price of risk, and margin constraints are all driven by the same state variable $s$, it is hard to disentangle the hedging demands produced by these factors individually. Nevertheless, the state-dependent constraints should be hedged: even a small probability that the margin constraint binds in the future changes investors’ current indirect utility function. As a result, the current optimal portfolio is affected.

**F. Equilibrium**

The equilibrium in the model is determined by the interaction between type A and type B investors. The definition of the equilibrium is standard: it implies that (i) all investors solve their utility maximization problem given market conditions and subject to their budget and margin constraints, (ii) aggregate consumption is equal to aggregate dividend, and (iii) all asset markets clear. For investors, the market conditions are described by the functions $r(s_t)$, $\mu_Q(s_t)$, $\sigma_Q(s_t)$, and $\bar{\omega}(s_t)$, and the state variable $s_t$ evolves according to equation (4) with drift $\mu_s(s_t)$ and diffusion $\sigma_s(s_t)$. All these functions are determined by equilibrium conditions.

To describe the equilibrium, I use two observations. First, because bonds are in net zero supply, at each moment investors of one type are lenders and investors of the other type are borrowers. Equation (8) implies that the optimal portfolio weight of the stock tends to be higher for less risk-averse investors (type B investors). Hence, it is reasonable to conjecture that in the equilibrium they are borrowers and may become constrained, whereas type A investors are always unconstrained. The latter assumption simplifies the analysis and, as demonstrated later, does indeed hold in the equilibrium.

Second, it is convenient to introduce the price–dividend ratio as a function of the state variable: $S_t/D_t = f(s_t)$. As demonstrated by Ang and Liu (2007),

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the expected returns, volatility of returns, and price–dividend ratio are closely related to each other. The exact result is stated in Lemma 1.

**Lemma 1:** The expected excess return on the risky asset \( \mu_Q(s) \) and the volatility of returns \( \sigma_Q(s) \) are determined by the state variable dynamics, the risk-free rate, and the price–dividend ratio of the risky asset:

\[
\mu_Q(s) = \mu_D - r(s) + \frac{1}{2} \sigma_s^2(s) \frac{f''(s)}{f(s)} + (\mu_s(s) + \sigma_D \sigma_s(s)) \frac{f'(s)}{f(s)} + \frac{1}{f(s)},
\]

(10)

\[
\sigma_Q(s) = \sigma_D + \frac{f'(s)}{f(s)} \sigma_s(s).
\]

(11)

**Proof:** See Appendix A.

Besides being useful for characterizing the equilibrium, Lemma 1 relates the volatilities of the state variable \( s \) and returns, and shows when the risky asset exhibits high volatility. Given that the price–dividend ratio \( f(s) \) is always positive, equation (11) implies that the volatility of returns \( \sigma_Q \) exceeds the volatility of dividends \( \sigma_D \) when \( f'(s) > 0, \sigma_s(s) > 0 \). This is a mild and natural condition and, as demonstrated below, is satisfied in the model.

The next proposition describes the equilibrium in the model.

**Proposition 2:** The equilibrium in the model is completely characterized by the functions \( r(s), \mu_s(s), \sigma_s(s), H(s), \) and \( f(s) \) that solve equations (A5) to (A9) in Appendix A. The expected excess return \( \mu_Q(s) \) and the volatility of returns \( \sigma_Q(s) \) are given by equations (10) and (11), respectively. The equilibrium market price of risk is given by equation (A10).

**Proof:** See Appendix A.

Inspection of equations (A5) to (A9) shows that the equilibrium is characterized by a system of differential equations for \( f(s) \) and \( H(s) \) and several algebraic equations for \( \mu_s(s), \sigma_s(s), \) and \( r(s) \). Note that the algebraic equations must be satisfied in each state \( s \). In general, \( \sigma_s(s) \) in equations (A5) and (A6) is determined by both \( f(s) \) and \( H(s) \) (see equation (A7), where \( \sigma_Q(s) \) is linked to \( f(s) \) as in equation (11)). This means that the differential equations for \( f(s) \) and \( H(s) \) are intertwined and should be solved simultaneously. In the special case in which the tightness of margin constraints \( \tilde{\omega} \) depends only on the volatility of returns and is inversely proportional to it, the volatility of the state variable

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8 A positive shock \( dB_t \) increases the aggregate dividend and consumption, and decreases the marginal utility of agents, so the market price of risk associated with it is positive (see equation (15) below). Thus, the variable is procyclical if it has a positive diffusion coefficient (e.g., \( \sigma_s(s) > 0 \) for the state variable \( s \)). Ito’s lemma implies that the price–dividend ratio is procyclical when \( f'(s) \sigma_s(s) > 0 \) or \( f'(s) > 0 \) when \( s \) is procyclical.
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σ_s(s) becomes unrelated to the price–dividend ratio (equation (A7) does not contain the price–dividend ratio \( f(s) \) because \( \bar{\omega}(s) \sigma_Q(s) = \text{const} \)). As a result, the differential equations for \( H(s) \) and \( f(s) \) can be disentangled: it is possible to solve the differential equation for \( H(s) \) and the algebraic equations first and then solve the differential equation for the price–dividend ratio \( f(s) \).

G. Equilibrium Volatility and Market Price of Risk

Although the equations characterizing the equilibrium in Proposition 2 look obscure, they still can be used to infer several important equilibrium properties. In particular, Proposition 2 makes it possible to compare the volatility of the state variable \( \sigma_s \) and the market price of risk \( \eta_s \) in constrained and unconstrained economies with identical fundamentals. The result is surprisingly unambiguous and is given by the following proposition:

**Proposition 3:** In the economy with margin constraints of a general form stated in equation (5),

1. the volatility of the state variable \( \sigma^c_s(s) \) in the constrained economy is not greater than the corresponding volatility \( \sigma^u_n(s) \) in the identical but unconstrained economy: \( \sigma^c_s(s) \leq \sigma^u_n(s) \) for all \( s \in [0, 1] \), and
2. the market price of risk \( \eta^c_s(s) \) in the constrained economy is not smaller than the corresponding market price of risk \( \eta^u_n(s) \) in the identical but unconstrained economy: \( \eta^c_s(s) \geq \eta^u_n(s) \) for all \( s \in [0, 1] \).

**Proof:** For \( s = 0 \) and \( s = 1 \), the statement is trivial since there is only one type of investor in such economies. In any other state \( s \in (0, 1) \), the volatility \( \sigma_s \) is determined by equation (A7), which can be rewritten as

\[
\sigma_D + \frac{\sigma_s}{s} = \min \left[ g(\sigma_s), \frac{\gamma_A}{\gamma_B} \left( \sigma_D - \frac{\sigma_s}{1-s} \right) \right],
\]

where \( g(\sigma_s) \) is a complicated function whose form depends on the form of the margin constraints \( \bar{\omega} \). Note that equation (12) is an algebraic equation for \( \sigma_s \), which should be satisfied in each state of the economy \( s \). Moreover, equation (A7) implies that the minimum in equation (12) is achieved on \( g(\sigma_s) \) when the constraint binds and on \( \frac{\gamma_A}{\gamma_B} \left( \sigma_D - \frac{\sigma_s}{1-s} \right) \) when the constraint is loose.\(^9\)

To prove the first statement of the proposition, assume that there exists a state of the economy \( s \) in which \( \sigma^c_s > \sigma^u_n \). Then equation (12) yields the following chain of inequalities:

\[
\sigma_D + \frac{\sigma^c_s}{s} = g(\sigma^c_s) \leq \frac{\gamma_A}{\gamma_B} \left( \sigma_D - \frac{\sigma^c_s}{1-s} \right) < \frac{\gamma_A}{\gamma_B} \left( \sigma_D - \frac{\sigma^u_n}{1-s} \right) = \sigma_D + \frac{\sigma^u_n}{s}.
\]

The first equality and first inequality hold because in the constrained case the minimum in equation (12) is achieved on \( g(\sigma_s) \). The second inequality follows

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\(^9\) This also explicitly follows from the derivation of equation (A7) presented in Appendix A.
from the assumption $\sigma_s^c > \sigma_s^{um}$. The last equality is again an implication of equation (12), but when the constraint is loose. The chain of inequalities implies that $\sigma_s^c < \sigma_s^{um}$, contradicting the assumption $\sigma_s^c > \sigma_s^{um}$ and thus completing the proof of the first statement in the proposition. The second statement immediately follows from the first one and equation (A10). Note that the proof is relatively simple because the volatility of the state variable in the unconstrained economy can be found in closed form and does not depend on indirect utilities. The latter results from market completeness in the absence of portfolio constraints. Q.E.D.

Proposition 3 states one of the main results of the paper. It should be emphasized that it is valid for a very general specification of margin constraints, which can depend nontrivially on the volatility of returns or be explicitly state dependent. As follows from the proof, the structure of the constraint only affects the function $g(\sigma_s)$, whose form is irrelevant for the result.

The reduction in the volatility of the state variable produced by portfolio constraints admits an intuitive explanation. In the unconstrained economy, the consumption shares of investors vary over time because investors dynamically share risks. This is a direct consequence of the heterogeneity in their preferences. Binding portfolio constraints reduce the difference between the portfolios held by type A and type B investors in the equilibrium, and this reduction brings the equilibrium variables closer to their counterparts in an economy with homogeneous agents where the consumption shares of agents are constant. Hence, the volatility of consumption shares declines.

Together with equation (11), Proposition 3 indicates that, when the price–dividend ratio is procyclical ($f(s)' > 0$), the volatility of returns in the constrained economy is not higher than in its unconstrained counterpart unless portfolio constraints substantially increase the semielasticity of the price–dividend ratio ($\log f(s)')'$ with respect to the state variable $s$. As I demonstrate below, the impact of portfolio constraints on the price–dividend ratio is quite weak and ($\log f(s)')'$ is almost the same in both economies. Thus, the volatility of returns is tightly linked to the volatility of the state variable and decreases in states in which margin constraints bind. Again, this is a very general conclusion that does not rely on the form of margin constraints. Hence, in contrast to conventional wisdom, binding margin constraints alone cannot produce an increase in volatility in general equilibrium in response to a negative shock to fundamentals. To explain such an increase, other market imperfections are required.

The increase in the market price of risk produced by binding margin constraints is also intuitive. In the unconstrained economy, type B investors hold a leveraged position in the risky asset that may be unattainable in the presence of margin constraints. If this is the case, the excessive supply of the risky asset is picked up by type A investors, who require compensation for holding additional stocks, so the market price of risk increases. Again, this is a very general effect that is not specific to a particular form of margin requirements.
II. Numerical Analysis

Although the decrease in the volatilities of the consumption share and returns and the increase in the market price of risk are established analytically, other properties of the constrained equilibrium described by Proposition 2 are not transparent. To obtain additional insights, I resort to numerical analysis.

A. Equilibrium without Margin Constraints

As a benchmark, consider a setting in which margin requirements are infinitely loose, that is, \( \bar{\sigma}(\sigma_s, \ldots) \to \infty \) for all \( s \). Without constraints, the model is essentially the same as in Bhamra and Uppal (2010) and Longstaff and Wang (2012), so I only briefly summarize the main properties of the equilibrium to facilitate its comparison with the equilibrium in a constrained economy.

In the absence of portfolio constraints, the structure of the equilibrium is much simpler. In particular, the drift and the volatility of the state variable \( s \) are given by explicit analytical formulas:

\[
\mu_s(s) = \left( \frac{\gamma_A}{1 - s} + \frac{\gamma_B}{s} \right)^{-1} \left( \gamma_A - \gamma_B \right) \left( \mu_D + \frac{1}{2} \sigma_D^2 \left( \frac{\gamma_A \gamma_B}{(\gamma_A s + \gamma_B(1 - s))^2 - 2} \right) - 2 \right),
\]  

(13)

\[
\sigma_s(s) = \sigma_D \left( \frac{\gamma_A}{1 - s} + \frac{\gamma_B}{s} \right)^{-1}.
\]  

(14)

Using equation (14) together with equation (A10), the market price of risk can be written as

\[
\eta(s) = \sigma_D \left( \frac{1 - s}{\gamma_A} + \frac{s}{\gamma_B} \right)^{-1}.
\]  

(15)

Moreover, the differential equations for the functions \( f(s) \) and \( H(s) \) in Proposition 2 decouple, and only the first one needs to be solved to find \( \mu_Q(s) \) and \( \sigma_Q(s) \). Longstaff and Wang (2012) provide a closed-form solution in terms of hypergeometric functions for the price–dividend ratio \( f(s) \) when the risk aversion of one group of investors is twice as high as the risk aversion of the other group. Chabakauri (2013) generalizes this result to models with arbitrary risk aversion parameters. Bhamra and Uppal (2010) provide a solution to a similar model in the form of a sum of an infinite series.

Equations (13) and (14) imply that the dynamics of the model are generated not only by fluctuations in the dividend, but also by wealth reallocation between agents with different degrees of risk aversion. As follows from equation (14), the volatility of the state variable is determined by the ratio \( \gamma_A / \gamma_B \), which characterizes the degree of heterogeneity between investors. Without heterogeneity in preferences, the consumption share of each agent is constant and the price of the risky asset is proportional to the current dividend.
Equation (15) shows that the market price of risk is solely determined by the aggregate relative risk aversion (Bhamra and Uppal (2010)).

To demonstrate the main properties of the equilibrium, I use a specific calibration of the model parameters. Following other studies (e.g., Kogan, Makarov, and Uppal (2007), Chabakauri (2013)), I calibrate the process for fundamentals $D_t$ using the historical mean and the volatility of the aggregate consumption growth rate in the United States. In particular, I set $\mu_D = 0.018$ and $\sigma_D = 0.032$. Also, I choose $\beta = 0.01$. To ensure that investor heterogeneity is large enough, I set $\gamma_A = 10$ and $\gamma_B = 2$. The latter choice also makes it possible to explore how hedging motives of type B investors affect the equilibrium.

To visualize the structure of the equilibrium, I plot several variables of interest as functions of the state variable $s$. The results are presented in Figure 1,
where a solid line corresponds to an unconstrained economy. Since $\sigma_s$ is positive (this immediately follows from equation (14)), the state variable $s_t$ is procyclical. The boundary points $s = 0$ and $s = 1$ correspond to economies dominated by investors of one type: when $s \to 0$ almost all dividends are consumed by type A, whereas when $s \to 1$ type B investors prevail.

Figure 1 shows that heterogeneity in preferences increases the volatility of the consumption share $s$, makes the interest rate $r$ and the market price of risk $\eta$ countercyclical, and makes the price–dividend ratio $S/D$ procyclical. As mentioned above, $\sigma_s$ would be zero in the absence of heterogeneity, so the first result is obvious. In an economy with homogeneous agents, the interest rate, market price of risk, and price–dividend ratio would be constant, whereas in Figure 1 the functions $r(s)$ and $\eta(s)$ are decreasing and the price–dividend ratio is increasing with $s$. Given that $s$ is procyclical, the rest of the statements follow immediately. Note that the slopes of $r$, $\eta$, and $S/D$ are determined by the dependence of these characteristics on the aggregate risk aversion, which decreases with $s$ as less risk-averse type B investors consume a bigger share of the dividend. In the limits $s = 0$ and $s = 1$, these characteristics are solely determined by type A and type B investors, respectively.

The middle-right panel of Figure 1 presents the volatility of returns $\sigma_Q$, which has a hump-shaped form. In the extremes $s = 0$ and $s = 1$, the volatility of returns coincides with the fundamental volatility $\sigma_D$ because the economy is populated by homogeneous agents and returns are driven only by innovations in cash flows. However, in the intermediate states, the volatility of returns is amplified by optimal risk sharing between heterogeneous investors. The expected returns $\mu_Q$ depicted in the upper-right panel also demonstrate a hump-shaped form, but in the majority of states they are countercyclical.

Figure 1 also shows the optimal weights of the risky asset in the portfolios of both types of investors. For type B investors, it is always greater than one and tends to one when $s \to 1$, whereas type A investors always hold a balanced portfolio of the stock and the risk-free asset. Thus, type B investors borrow from type A investors and invest the loan along with their own wealth in the risky asset. Interestingly, the optimal leverage of type B investors is higher in bad states when investment opportunities are especially attractive from their perspective. This means that, when imposed, margin constraints are likely to bind in bad states of the economy.

B. Equilibrium with Margin Constraints

To explore the impact of margin constraints on the equilibrium prices and returns, I again use the setting from Section I but now investors are subject

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10 The form of $\sigma_Q$ as a function of $s$ may be different for alternative parameter values. In particular, it may take a U-shaped form, meaning that in many states the volatility of returns is lower than the volatility of fundamentals (Longstaff and Wang (2012)). Bhamra and Uppal (2009) identify the exact condition that determines whether the volatility of returns is larger or smaller than the fundamental volatility.
to margin requirements. Unfortunately, the system of nonlinear differential equations from Proposition 2 is rather complex and does not admit a closed-form solution. Therefore, I have to rely on numerical techniques. To find a numerical solution, I use the projection method, which approximates the functions $H(s)$ and $f(s)$ by linear combinations of Chebyshev polynomials (Judd (1998)). One of the advantages of this approach is that the solutions are automatically twice continuously differentiable for all $s \in (0, 1)$. Details on the implementation of the projection method are provided in Appendix B.

To make the results comparable to those from the previous section, I consider an economy with the same fundamentals as before and the same heterogeneity across agents. Since in practice margin requirements are often determined by the volatility of stock returns, I assume that

$$\bar{\omega}(\sigma_Q(s)) = \bar{m}\left(\frac{\sigma_D}{\sigma_Q(s)}\right)^\alpha, \quad (16)$$

where $\alpha$ and $\bar{m}$ are parameters of the constraint. The parameter $\bar{m}$ determines the average tightness of margin requirements, and I set $\bar{m} = 1.3$. Because $\bar{m} > 1$, investors are unconstrained in at least some states of the economy with low volatility of returns. However, the constraint is sufficiently tight, so that there are states of the economy in which it binds type B investors.

The constraint in the form of equation (16) encompasses two interesting cases that I focus upon. When $\alpha = 0$, margin requirements are constant. When $\alpha = 1$, equation (16) gives an approximation to margin requirements determined by the value-at-risk (VaR) rule (Danielsson, Shin, and Zigrand (2004)) and effectively represents a constant constraint imposed on the maximal permitted volatility of investor wealth. Note that, in the latter case, the level of margin constraints in equation (16) is endogenous because it is related to the volatility of returns, which is determined in the equilibrium.

Figure 1 shows various equilibrium characteristics in the constrained economy along with their counterparts in the unconstrained case. The first important observation from Figure 1 is that qualitatively the impacts of both types of constraints are similar. The only difference is in magnitudes: since the volatility of returns $\sigma_Q$ is higher than the volatility of fundamentals $\sigma_D$, the time-varying margins from equation (16) with $\alpha = 1$ are more restrictive on average than constant margins. As a result, all effects are stronger when $\alpha = 1$.

The middle-left panel of Figure 1 depicts $\sigma_s$ and illustrates the first statement of Proposition 3: portfolio constraints of both types reduce the volatility of the state variable. Because the impact of the constraints on the sensitivity of the price–dividend ratio to the state variable is very small, a decrease in $\sigma_s$ translates into a decrease in the volatility of returns $\sigma_Q$ presented in the middle-right panel of Figure 1 (this follows from equation (11)). The graph of $\sigma_Q$ also shows that, although the exact point at which the constraint starts binding is determined by the form of the constraint, the volatility decreases only in the states where the constraint actually binds. This is an implication of equation (A7).
Figure 1 also demonstrates that portfolio constraints tend to increase the market price of risk and reduce the risk-free rate. The former effect illustrates the second statement of Proposition 3. The decrease in the risk-free rate produced by margin constraints is also intuitive. Without constraints, type B investors borrow from type A investors and build leveraged portfolios. If maximum leverage is restricted, the demand for credit is lower and, as a result, the equilibrium risk-free rate is also lower. This is a general effect, which does not depend on the form of the constraints. These two effects are reported by Kogan, Makarov, and Uppal (2007) for an economy with constant borrowing constraints and less risk-averse investors, who have logarithmic preferences. My results indicate that the conclusions of Kogan, Makarov, and Uppal (2007) generalize to other forms of portfolio constraints and to economies with arbitrary levels of risk aversion.

In the majority of states, binding portfolio constraints tend to increase expected excess returns $\mu_Q$. This is also intuitive: because portfolio holdings of less risk-averse investors are constrained in some states, more risk-averse investors are forced to hold more risky assets than they would without constraints. To induce them to buy additional stocks, the risk premium should be higher. In those states in which constraints decrease expected excess returns, the risky asset is more attractive due to a substantially lower volatility resulting in a higher market price of risk.

The left panel in the bottom row of Figure 1 shows the impact of margin constraints on the price–dividend ratio. Although it is widely believed that constraints decrease asset valuations, my results indicate that this is not the case in general equilibrium: when margins bind, the price–dividend ratio increases. Intuitively, this effect can be attributed to the lower risk-free rate. Even though constraints increase the risk premium, total expected returns decrease because the lower risk-free rate cannot be offset by the higher excess expected returns. However, the magnitude of the effect is small: the maximum increase in the price–dividend ratio across all states is only around 5%.

Finally, Figure 1 shows the effect of constraints on optimal portfolio weights. As expected, type B investors in all states of the economy hold a leveraged portfolio, but leverage is restricted in many states and does not exceed $\bar{m} = 1.3$. Correspondingly, type A agents invest more heavily in stocks to absorb the extra supply, but still always provide margin credit to type B investors. Hence, margin requirements affect type A investors only indirectly through the changed investment opportunities (expected returns, volatility, etc.).

C. Equilibrium with Myopic Type B Investors

So far, I assume that type B investors have CRRA preferences over their consumption stream. Since investment opportunities are state dependent in

11 Under certain conditions, equilibrium prices in economies with portfolio constraints may even contain rational bubbles (Hugonnier (2012)). In the case $\alpha = 1$, Prieto (2011) shows that the stock price is free of bubbles if more risk-averse investors are always unconstrained and less risk-averse investors are unconstrained in some states of the economy.
the equilibrium, the optimal portfolio of type B investors contains both myopic and hedging components when $\gamma_B \neq 1$, and their consumption–wealth ratio is also state dependent. In this section, I explore the impact of the hedging demand on the properties of the equilibrium. For this purpose, I compare the fully rational equilibrium discussed in the previous section with an equilibrium in a modified model in which type B investors are assumed to be myopic. To construct the latter equilibrium, it is sufficient to set $H = \text{const}$ in equations (A5), (A7), (A8), and (A9). This eliminates the hedging demand and makes the consumption–wealth ratio constant.

To visualize the effect of hedging, I plot the ratios of various characteristics in the equilibria with rational and myopic type B investors. Again, I consider an unconstrained economy along with constrained ones with margin requirements from equation (16). The results are presented in Figure 2.

The first interesting observation from Figure 2 is that the hedging demand in the unconstrained economy has no impact on the equilibrium statistics (all ratios are equal to one) even though in some states the hedging component accounts for almost 20% of the myopic demand (see the bottom-right panel of Figure 2). This is a consequence of market completeness for all investors and homothetic preferences.\footnote{The same combination of properties made it possible to find explicit formulas for $\mu_s$, $\sigma_s$, and $\eta$ in Section II.A.} Note that this property also holds in economies with portfolio constraints, but only in those states in which these constraints do not bind.

Next, Figure 2 shows that hedging of the time variation in investment opportunities by type B investors changes the equilibrium characteristics when portfolio constraints bind, and the effect is strong. In the unconstrained region, investment opportunities are worse in good states, so the hedging demand is positive and increases the leverage desired by type B investors. As a result, the presence of hedging demand expands the set of states in which type B investors hit the margin constraint. These are the states where the impact of hedging is particularly strong. However, even in those states in which type B investors are constrained both with and without hedging demand, the equilibrium statistics are affected by the hedging component.

As before, the effects of constant and time-varying constraints are qualitatively similar. Also, comparing Figures 1 and 2 it is easy to see that portfolio constraints change the equilibrium variables in the same direction in economies with and without hedging demand, but hedging amplifies the impact of constraints. In particular, the presence of the hedging component substantially reduces the volatility of the state variable as well as its drift when the constraint binds. This effect can be explained by better consumption smoothing of type B investors when the consumption–wealth ratio is allowed to be state dependent.

Overall, my analysis reveals that the hedging motive of potentially constrained investors may amplify the impact of portfolio constraints, so models
Figure 2. The impact of hedging demand of type B investors on the equilibrium in the presence of portfolio constraints. This figure presents ratios of various variables in the equilibria with fully rational and myopic preferences of type B investors as functions of the state variable $s$ (consumption share of type B investors). $\mu_s$ is the drift and $\sigma_s$ is the volatility of the state variable $s$, $r$ is the risk-free rate, $\eta$ is the market price of risk, $\mu_Q$ is the drift and $\sigma_Q$ is the volatility of excess returns on the risky asset, $S/D$ is the price–dividend ratio, and $\omega^A$ and $\omega^B$ are the portfolio weights of the risky asset held by type A and type B investors, respectively. The model parameters are as follows: $\mu_D = 0.018$, $\sigma_D = 0.032$, $\beta = 0.01$, $\gamma_A = 10$, and $\gamma_B = 2$. The margin constraints are specified by equation (16) with $\overline{m} = 1.3$. The solid line corresponds to an unconstrained equilibrium, the dashed line corresponds to an equilibrium with a constant portfolio constraint ($\alpha = 0$), and the dotted line corresponds to an equilibrium with a VaR-type constraint ($\alpha = 1$).

with logarithmic investors may not adequately capture the magnitude of the effects.

**III. Extension: Two Risky Assets**

So far, I assume that the economy contains only one risky asset. In this section, I extend the analysis to an economy with two Lucas trees and explore how the imposition of margin requirements changes the volatilities of the assets and
the correlation between their returns. I assume that the dividends of risky assets $i = 1, 2$ follow geometric Brownian motions with identical parameters:

$$\frac{dD_{it}}{D_{it}} = \mu_D dt + \sigma_D dB_{it}. \quad (17)$$

To avoid unnecessary complications, the innovations of the standard Brownian motions $dB_{1t}$ and $dB_{2t}$ are assumed to be uncorrelated. In matrix notation, they form a vector $dB_t = [dB_{1t} \ dB_{2t}]'$. The excess returns on the risky assets evolve according to

$$dQ_t = \mu_Q dt + \Sigma_Q dB_t, \quad (18)$$

where the $2 \times 1$ matrix of risk premia $\mu_Q$ and the $2 \times 2$ matrix of diffusions $\Sigma_Q$ are determined in the equilibrium.

As before, one of the state variables is the consumption share of type B agents $s_t = C^B_t / (D_{1t} + D_{2t})$, which evolves according to

$$ds_t = \mu_s dt + \Sigma_s dB_t, \quad (19)$$

where $\mu_s$ and $\Sigma_s$ are determined by equilibrium conditions. Following the literature on exchange economies with multiple Lucas trees (e.g., Menzly, Santos, and Veronesi (2004), Cochrane, Longstaff, and Santa-Clara (2008), Martin (2013)), the other state variable is chosen to be the first asset’s share in the aggregate dividend: $u_t = D_{1t} / D_t$, where $D_t = D_{1t} + D_{2t}$. Using $u_t$, the dynamics of the aggregate dividend can be described as

$$\frac{dD_t}{D_t} = \mu_D dt + \Sigma_D(u_t) dB_t, \quad (20)$$

where $\Sigma_D(u) = [\sigma_D u \ \sigma_D (1 - u)]$. The stochastic equation for $u_t$ immediately follows from equation (17),

$$du_t = \mu_u(u_t) dt + \Sigma_u(u_t) dB_t, \quad (21)$$

where

$$\mu_u(u) = \sigma_u^2 u(1 - u)(1 - 2u), \quad \Sigma_u(u) = [\sigma_D u(1 - u) - \sigma_D u(1 - u)]. \quad (22)$$

Note that the dynamics of $u_t$ are fully determined by model assumptions, so only the evolution of $s_t$ is an equilibrium outcome, just as in the model with a single Lucas tree.

Again, I assume that there are two types of CRRA agents (type A and type B) with different coefficients of risk aversion. For the sake of tractability, type B agents are assumed to have logarithmic preferences. Although the latter

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13 A model with multiple trees and margin constraints is also considered in Garleanu and Pedersen (2011). However, the authors focus on changes in expected returns and asset valuations produced by constraints, whereas my objective is to study the volatilities and the correlations of returns.
assumption eliminates the hedging demand of type B investors, it substantially simplifies the characterization of the equilibrium.

The key ingredient of the model is margin requirements imposed on agents. In the model with one risky asset, the requirements constrain only long-leveraged positions (effectively, short positions in the risk-free asset). Although constraints on short positions in the stock could also be imposed, they are irrelevant in the equilibrium because no one wants to short a risky asset. The situation is drastically different when there are multiple stocks in the economy. If margin requirements are imposed only on long positions, agents can circumvent them by shorting one of the stocks instead of the risk-free asset. Although in such a case the equilibrium properties would be different from those in the unconstrained economy, the lack of margin requirements on the short side is unrealistic. This point is emphasized in Brunnermeier and Pedersen (2009) and Garleanu and Pedersen (2011). These papers argue that it is more appropriate to assume that the constraints are imposed on both long and short positions and have the form

\[
\sum_{i=1}^{2} (m_i^\omega_i^+ + m_i^\omega_i^-) \leq 1, \tag{23}
\]

where \(\omega_i, i = 1, 2\), are portfolio weights of risky assets, and \(\omega_i^\pm = \max(\omega_i, 0)\) and \(\omega_i^- = \max(-\omega_i, 0)\) are long and short positions, respectively. In general, the parameters of the constraints \(m_i^\omega_i^+\) and \(m_i^\omega_i^-\) can be state dependent and determined by various endogenous variables like the volatility of returns. Effectively, equation (23) imposes four linear constraints on portfolio weights that can be written as

\[
\omega^\prime m^K \leq 1, \quad K = 1, \ldots, 4, \tag{24}
\]

where \(\omega = [\omega_1 \, \omega_2]^\prime\) and \(m^1 = [m_1^\omega_1^+ \, m_2^\omega_2^+]^\prime\), \(m^2 = [m_1^\omega_1^- \, m_2^\omega_2^+]^\prime\), \(m^3 = [m_1^- \, m_2^\omega_2^-]^\prime\), and \(m^4 = [m_1^- \, m_2^-]^\prime\). Motivated by this observation, I assume that the margin requirements have the form of equation (24) with

\[
m^K = \Sigma Q A^K, \tag{25}
\]

where

\[
A^1 = \frac{1}{\bar{m}\sigma_D} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A^2 = \frac{1}{\bar{m}\sigma_D} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad A^3 = \frac{1}{\bar{m}\sigma_D} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad A^4 = \frac{1}{\bar{m}\sigma_D} \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \tag{26}
\]

and \(\bar{m}\) is a constant that determines the average tightness of the constraints.

The suggested specification of margin requirements deserves several comments. First, it represents a generalization of the one-asset constraint from equation (5) with the threshold from equation (16) and \(\alpha = 1\). As demonstrated in Section I, the equation for \(\sigma_s(s)\) simplifies when the tightness of constraints is inversely proportional to volatility, and the two differential equations that
describe the equilibrium dynamics can be solved separately. A very similar simplification occurs in the multiasset case, and, together with the assumption of logarithmic preferences, it results in a closed-form solution for $\Sigma_s$ (see Proposition 4). This makes the model more tractable.

Second, margin requirements have the same functional form for both assets. Although it is easy to incorporate cross-sectional heterogeneity of margin requirements in the model, I deliberately avoid it to ensure that all cross-sectional effects are purely endogenous and not produced by the assumed differences in margins. Note that in the vast majority of states the tightness of margin constraints is not the same for the two assets because it is determined by asset volatilities, which vary across stocks.

Third, the constraint in the form of equation (25) implies that the exposures of the portfolio to the individual shocks $dB_1$ and $dB_2$ are collateralized separately. When returns are uncorrelated, the matrix $\Sigma_Q$ is diagonal and separate collateralization of shocks reduces to separate collateralization of individual assets.

The definition of the equilibrium is standard and mimics its analog from Section I: it states that (i) all investors maximize their utilities subject to the budget and margin constraints, taking investment opportunities as given, (ii) aggregate consumption is equal to aggregate dividend, and (iii) all asset markets clear. Because type B investors are less risk averse, they are likely to use leverage in the equilibrium and may be constrained by margin requirements in some states of the economy. Moreover, it is conceivable that type A investors may also be bound by margin constraints when the equilibrium properties of the risky assets are so different that these investors want to short one asset and invest more in the other one. In this case, the market becomes incomplete for all investors, which substantially complicates the characterization of the equilibrium.

To preserve tractability, I consider only those equilibria in which type A investors are unconstrained in all states of the economy. Although such an equilibrium may not exist for some extreme combinations of model parameters, this is unlikely to limit the generality of my analysis. The constraint can bind type A investors if one of their portfolio weights is highly negative. For this to happen, either a myopic component or a hedging component should be negative. However, the former possibility is ruled out because in this case type B investors would also short the same asset (due to their logarithmic preferences, their asset demand contains only a myopic component) and no one would hold that asset in the equilibrium. Hence, a negative portfolio weight of type A investors can be produced only by the hedging component. Although it is hard to characterize the relative size of the hedging demand in general (it is determined by derivatives of marginal utility with respect to state variables), for realistic parameters it tends to be smaller than the myopic demand, so the total demand is likely to be positive. This makes the constraint irrelevant for type A investors. In the numerical example presented below, I find the

14 I am very grateful to an anonymous referee for pointing this out.
equilibrium assuming that type A investors are unconstrained and then confirm that their optimal portfolios do indeed satisfy margin requirements.

Proposition 4 formally describes the equilibrium in the model.

**Proposition 4:** If an equilibrium in which type A investors are unconstrained in all states of the economy exists, it is characterized by (i) the drift $\mu_s(s, u)$ and diffusion $\Sigma_s(s, u)$ of the endogenous state variable $s$, (ii) the risk-free rate $r(s, u)$ and market price of risk $\eta(s, u)$, and (iii) the price–dividend ratio functions $S_1/D_1 = f_1(s, u)$ and $S_2/D_2 = f_2(s, u)$. The functions $\mu_s(s, u)$, $\Sigma_s(s, u)$, $r(s, u)$, and $\eta(s, u)$ are given by equations (A23) to (A27) in Appendix A. The functions $f_1(s, u)$ and $f_2(s, u)$ are solutions to the second-order partial differential equations (A27). The expected excess returns $\mu_Q(s, u)$, the diffusion of returns $\Sigma_Q(s, u)$, and the optimal portfolio weights of type B investors $\omega^B(s, u)$ are algebraically linked to the functions characterizing the equilibrium as stated in equations (A28) to (A30). The optimal portfolio weights of type A investors $\omega^A(s, u)$ are given by equation (A31).

**Proof.** See Appendix A.

Although analytic formulas are available for many variables, the properties of the equilibrium are more evident when all variables are plotted as functions of $u$ and $s$. To facilitate comparison of the results with those in a one-asset economy, I keep the parameters the same as before (except $\gamma_B = 1$ instead of $\gamma_B = 2$).

As in Section II, the economy without margin requirements serves as a benchmark, and its equilibrium variables are presented in Figure 3. To save space, I plot several characteristics (expected returns, volatility of returns, beta, portfolio weights) for the first asset only. Because the assets have identical fundamentals and the state variable $u$ measures the contribution of the first asset to the aggregate dividend, the plots for the second asset can be obtained from the plots for the first one as a mirror image with respect to the plane $u = 1/2$ (i.e., by substituting $1 - u$ for $u$ in all formulas).

The relations between the majority of equilibrium characteristics and the state variable $s$ resemble the patterns observed in the one-asset case (see Figure 1), and as $u \to 1$, all variables converge to their counterparts in a one-asset economy, in which only the first risky asset exists. As before, investor heterogeneity makes returns more volatile than fundamentals (recall that the volatility of dividends is $\sigma_D = 0.032$) and gives the risk-free rate and market price of risk a downward-sloping shape.

Figure 3 also demonstrates how the equilibrium statistics depend on the relative size of the first asset $u$, and shows that many observed relations appear to be similar to those identified by Cochrane, Longstaff, and Santa-Clara (2008) in an economy with two Lucas trees but homogeneous investors. In particular, the expected excess returns, the volatility of returns, and the volatility of the state variable $s$ have a U-shaped form, whereas the risk-free rate has a hump-shaped form for any fixed $s$. These patterns are explained by better diversification opportunities available to investors when trees have comparable
Figure 3. The equilibrium with two risky assets (no margin constraints). This figure presents various variables in the unconstrained equilibrium in an economy with two risky assets and heterogeneous agents. All variables are functions of the consumption share $s$ of type B investors and the share $u$ of the first dividend $D_1$ in the aggregate dividend $D$. $\mu_s$ is the drift and $\sigma(ds)$ is the conditional volatility of the state variable $s$, $r$ is the risk-free rate, $\eta_1$ is the market price of risk of the shock $dB_1$, $\mu_{Q1}$ is the drift and $\sigma(dQ_1)$ is the volatility of excess returns on the first risky asset, $S_1/D_1$ is the price–dividend ratio of the first asset, $\omega^A_1$ and $\omega^B_1$ are the portfolio weights of the first risky asset held by type A and type B investors, $\beta_1$ is the market beta of the first asset, and $\rho(dQ_1, dQ_2)$ is the conditional correlation between stock returns. The model parameters are as follows: $\mu_D = 0.018$, $\sigma_D = 0.032$, $\beta = 0.01$, $\gamma_A = 10$, and $\gamma_B = 1$.

size, that is, when $u$ is close to $1/2$. As $u$ approaches one, the relative size of the first asset as well as the risk associated with it increases, which leads to a higher market price of risk $\eta_1$. Figure 3 also shows that the weights of the first asset in the portfolios of type A and type B investors grow with the relative
size of the asset. This is intuitive because investors have to put more wealth in the first asset in the absence of a sufficient amount of the second one as \( u \to 1 \). Finally, note that the impact of agents’ heterogeneity has a stronger impact than the presence of two trees. In particular, diversification makes returns less volatile than fundamentals as in Cochrane, Longstaff, and Santa-Clara (2008), but in the majority of states this effect is offset by the increase in the volatility produced by heterogeneity in investor risk preferences.

Note that there are two new variables that do not appear in the one-asset model: the market beta of the first asset \( \beta_1 \) and the correlation between returns on the risky assets. The bottom-right panel of Figure 3 shows that the correlation is positive in all states of the economy, even though the assets have independent fundamentals. This pattern obtains for two reasons. First, the correlation between returns is positive even in an economy with two trees but homogeneous investors (Cochrane, Longstaff, and Santa-Clara (2008)). The intuition of this effect is relatively straightforward: a negative shock to dividends on the first asset lowers its price and increases the relative share of the second asset. Because investors want to maintain the optimal diversification of their portfolios, they start selling the second asset. As a result, its price also decreases and stock returns become positively correlated.

Second, heterogeneity in investor preferences contributes to the positive correlation of returns. A negative shock to the dividend on the first asset decreases the price of that asset as well as investors’ wealth and causes a reallocation of risky assets from type B to type A investors (this effect is also present in the one-asset case). Hence, the aggregate risk aversion of investors in the economy increases, and the prices of all risky assets, including the second asset, fall. As a result, returns on the assets become positively correlated. Together with the increase in the volatility of returns, this is another consequence of risk sharing among heterogeneous investors and the wealth effect. In the majority of states, the correlation between returns increases as the share of type B investors decreases, that is, it is a decreasing function of \( s \).

The bottom-left panel of Figure 3 shows that the beta of the first asset \( \beta_1 \) is largely determined by two factors: the correlation of the assets and the relative size of the first asset. In particular, the beta is close to one when the assets are highly correlated, which happens when \( s \) is relatively low. Also, \( \beta_1 \) is high when the first asset is large and constitutes a substantial share of market value (i.e., \( u \) is close to one). In the limit \( u = 1 \), there is only one asset in the economy, and its beta mechanically equals one.

To illustrate the impact of margin requirements, I consider the same economy as before but with the constraints specified by equations (24) to (26). For consistency with the analysis of the one-asset economy, I set \( \bar{m} = 1.3 \). Figure 4

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15 The market betas of stocks are computed as \( \beta_i = \frac{\text{Cov}_t(dQ_{it}, dQ_{mt})}{\text{Var}_t(dQ_{mt})} \), where the market excess return can be represented as \( dQ_{mt} = w_1dQ_{1t} + w_2dQ_{2t} \), and \( w_1 = u f_1/(u f_1 + (1 - u) f_2) \) and \( w_2 = (1 - u) f_2/(u f_1 + (1 - u) f_2) \).

16 Contagion caused by the wealth effect has also been studied by Kyle and Xiong (2001) and Ehling and Heyerdahl-Larsen (2012), among others.
Figure 4. The equilibrium with two risky assets and margin constraints. This figure presents various variables in the economy with two risky assets, heterogeneous agents, and margin constraints. All variables are functions of the consumption share $s$ of type B investors and the share $u$ of the first dividend $D_1$ in the aggregate dividend $D$. $\mu_s$ is the drift and $\sigma(ds)$ is the volatility of the state variable $s$, $r$ is the risk-free rate, $\eta_1$ is the market price of risk of the shock $d\beta_1$, $\mu_{Q_1}$ is the drift and $\sigma(dQ_1)$ is the conditional volatility of excess returns on the first risky asset, $S_1/D_1$ is the price–dividend ratio of the first asset, $\omega_A^1$ and $\omega_B^1$ are the portfolio weights of the first risky asset held by type A and type B investors, $\beta_1$ is the market beta of the first asset, and $\rho(dQ_1, dQ_2)$ is the conditional correlation between stock returns. The model parameters are as follows: $\mu_D = 0.018$, $\sigma_D = 0.032$, $\beta = 0.01$, $\gamma_A = 10$, $\gamma_B = 1$, and $\bar{m} = 1.3$.

shows the same variables as Figure 3, but in the economy with time-varying margin requirements.

Comparison of Figures 3 and 4 yields several important observations. First, margin requirements decrease the volatility of the state variable $s$ as well as
the volatility of stock returns in all states of the economy where they bind (this also follows from equations (A23) and (A28)). The intuition for this result is the same as for its one-asset analog: margin requirements force type B investors to hold less leveraged portfolios, so the difference in portfolios of type A and type B investors becomes smaller, and the excessive volatility produced by investor heterogeneity decreases.

A second and more interesting observation is that the magnitude of the volatility reduction varies across states of the economy. When the trees have comparable size (these states have $u$ close to $1/2$ and belong to the region $A_1$ defined in equation (A23)), the graph of $\sigma(ds)$ has a saddle shape that resembles the shape of the volatility graph in the unconstrained economy (see Figure 3), but with a lower average volatility. As follows from the proof of Proposition 4, only the borrowing constraint binds type B investors in those states. However, two constraints bind in states of the economy with low values of $s$ and relatively low or high values of $u$ (these states belong to the regions $A_{12}$ and $A_{13}$ defined in equation (A23)). This exacerbates the decline in volatility, which appears to be stronger for more extreme values of $u$, and explains why the “valley” graph for $\sigma(ds)$ in Figure 4 has additional “slope cuts” compared to its analog in Figure 3.

The effect is qualitatively different for the volatility of returns. The middle-right panel of Figure 4 implies that the returns on a large asset (e.g., the first asset when $u$ is close to one) are more volatile than fundamentals, whereas the volatility of returns on a small asset (e.g., the first asset when $u$ is close to zero) can be even lower than the volatility of fundamentals. Given that at almost any moment one asset is large and the other asset is small, margin constraints produce strong heterogeneity in asset volatilities in the cross-section. Note that this pattern of volatilities is in stark contrast with what happens without constraints: the middle-right panel of Figure 3 shows that the graph for volatility is almost symmetric relative to the plane $u = 1/2$, implying that both stocks at each moment have very similar volatilities.

To understand the intuition behind the dispersion of volatilities, consider the graph for the market beta of the first asset $\beta_1$ presented in the bottom-left panel of Figure 4. It shows that the large asset (the first asset when $u$ is high) has a much higher beta than the small one (the first asset when $u$ is low). This difference results from two effects. First, the asset with a higher dividend represents a larger share of the market, so its beta is closer to one almost mechanically. This is observed in both constrained and unconstrained economies (see Figure 3). Second, margin constraints decrease the correlation of returns (as I discuss below), thereby further increasing the gap in betas.

As in the one-asset economy, a less risk-averse investor prefers to hold a leveraged portfolio. Without margin constraints, this portfolio is constructed by shorting a risk-free asset and investing the proceeds in a diversified portfolio of the risky assets. However, such a portfolio is unattainable in the presence of constraints. Instead, it becomes optimal to overweight the asset with the highest beta, sacrificing the benefits of diversification for the sake of having a
portfolio with the highest possible leverage.\footnote{A similar effect is discussed by Frazzini and Pedersen (2013) in the context of the impact of constraints on expected returns.} This is clearly illustrated by the panel of Figure 4 that presents the weight of the first asset $\omega^B_1$ in the portfolio of type B investors (again, the weight of the second asset $\omega^B_2$ can be obtained from the graph for $\omega^B_1$ by flipping it around the plane $u = 1/2$): in the majority of states, type B investors build a leveraged position by using only the risky asset that has the largest dividend share. For example, when $u$ is large (the first tree dominates the second one), the portfolio weight of the second asset is positive but close to zero, and almost all capital is invested in the first asset. Given that type B investors nearly ignore the smallest asset (e.g., the first asset when $u$ is substantially below 1/2), investor heterogeneity plays little role in amplifying its volatility. As a result, the diversification effect dominates, and the returns on the smallest asset can be even less volatile than fundamentals. However, the increase in volatility produced by investor heterogeneity and risk sharing is unaffected for the large asset, and this explains why the volatilities of returns may be so different in the cross-section in the presence of margin constraints.

Figure 4 also presents the graph for the weight of the first asset $\omega^A_1$ in the portfolio of type A investors. In contrast to the weights of type B investors, type A investors always maintain unleveraged long positions in both assets, which means that type A investors never hit margin constraints in the equilibrium even though these constraints have been imposed on them. This result justifies the construction of the equilibrium using the assumption that type A investors are unconstrained in all states of the economy.

Finally, margin requirements have a strong effect on the correlation between asset returns. Comparing Figures 3 and 4, it is easy to see that the correlation in all states decreases relative to the unconstrained economy, and this diminishes the asset beta when the asset constitutes a relatively small fraction of the market. Moreover, the decline in the correlation is more pronounced when the trees have comparable size ($u$ is around 1/2). This effect has an intuitive explanation. When there is a positive shock to the dividend on the first asset, its price increases. Moreover, the share of the first asset $u$ increases, so, according to Figure 4, the optimal weight of this asset in the portfolio of type B investors also increases. Hence, there is selling pressure on the second asset, which drives its price down. Because the prices move in opposite directions, the correlation between assets declines.

**IV. Conclusion**

To summarize, this paper presents a novel theoretical analysis of an exchange economy with heterogeneous investors and state-dependent margin requirements. In contrast to existing studies, which mostly focus on constant margin requirements, I assume that margins are time varying and their tightness is determined endogenously in the equilibrium. This model delivers several new insights: (i) binding margin constraints reduce the risk-free rate as
well as the volatility and correlation of returns, but increase expected returns, the market price of risk, and the price of the risky asset, (ii) although some of these effects have been documented in the literature for constant portfolio constraints, my analysis shows that their validity is much broader and that they hold even when margin constraints depend on market conditions, (iii) because the volatility of returns unambiguously decreases when margin constraints bind, only the interaction between margins and other market frictions absent in the model can produce the often observed increase in volatility when some investors are constrained, (iv) hedging of time-varying investment opportunities by constrained investors may have a strong quantitative effect on the properties of the equilibrium, and (v) margin constraints may produce strong cross-sectional dispersion of stock return volatilities even when the constraints are ex ante the same for all assets. Although the majority of the results are interesting mostly from a theoretical standpoint, the last result implies that the introduction of margin constraints increases the cross-sectional dispersion of volatilities. This is a new prediction, and testing it would be an interesting topic for future empirical research.

The model developed in the paper can be generalized in numerous ways. First, investors are assumed to be price takers. However, if the amount of the risky asset that they can buy or sell is nonnegligible, price impact may become an important issue. In this case, investors may start worrying about the liquidity of the market in addition to its volatility, and liquidity may become another important determinant of margin requirements.

Second, the paper assumes that the functional form of the relation between margin requirements and current market conditions is given exogenously. However, the ability of lenders to change margin requirements may open a door for their strategic interaction. In particular, lenders may inefficiently coordinate on price volatility or engage in predation against borrowers in some states of the economy. Understanding the strategic behavior of lenders and its general equilibrium implications is likely to be a fruitful direction for future research.

Appendix A: Proofs

Proof of Proposition 1: Because in this proof I consider investors of one type only, I omit the investor subscript to simplify notation. All functions depend on \( s_t \) only, and this argument is also omitted. The value function of an investor who maximizes the CRRA utility function satisfies the following Bellman equation:

\[
\max_{\{C, \omega \leq \bar{\omega}\}} \left[ e^{-\beta t} u(C) + DJ \right] = 0, \tag{A1}
\]

where

\[
DJ = JW(\omega W \mu_Q + r W - C) + \frac{1}{2} J_{WW} W^2 \omega^2 \sigma_Q^2 + J_s \mu_s + \frac{1}{2} J_{ss} \sigma_s^2 + J_{W_s} \omega \sigma_Q \sigma_s + J_t.
\]
Due to homotheticity of preferences, it is natural to look for the value function in the following standard form:

\[ J(s, W, t) = \frac{1}{1-\gamma} W^{1-\gamma} \exp(H) \exp(-\beta t), \tag{A2} \]

where the function \( H(s) \) is assumed to be twice continuously differentiable. The maximization in equation (A1) over \( C \) yields

\[ e^{-\beta t} u'(C) - J_W = 0 \quad \text{or} \quad C = W \exp\left(-\frac{H}{\gamma}\right). \tag{A3} \]

The constrained maximization over \( \omega \) reduces to

\[ \omega = \arg \max_{\omega \leq \bar{\omega}} \left[ (J_W \mu_Q + J_W s \sigma_Q s) \omega + \frac{1}{2} J_{WW} W \sigma_Q^2 \omega^2 \right]. \]

Given that \( J_{WW} < 0 \), the solution to the maximization problem is either an internal point satisfying the first-order condition or a point on the boundary \( \omega = \bar{\omega} \). Thus, solving the first-order condition and using equation (A2),

\[ \omega = \min \left( \bar{\omega}, \frac{\mu_Q}{\gamma \sigma_Q^2} + \frac{\sigma_s \gamma \sigma_Q H'}{\gamma \sigma_Q} \right). \tag{A4} \]

Substitution of the optimal consumption (A3) and the optimal portfolio policy (A4) along with the value function (A2) into equation (A1) yields equation (9).

Proof of Lemma 1: By definition of the price–dividend ratio, the price of the risky asset is \( S_t = D_t f(s_t) \). Applying Ito’s lemma,

\[ \frac{dS_t}{S_t} = \frac{dD_t}{D_t} + \frac{df(s_t)}{f(s_t)} + \frac{dD_t}{D_t} \frac{df(s_t)}{f(s_t)}, \]

where

\[ df(s_t) = f'(s_t)(\mu_s(s_t) dt + \sigma_s(s_t) dB_t) + \frac{1}{2} f''(s_t) \sigma_s(s_t)^2 dt. \]

Using the structure of the dividend process from equation (1) and performing some algebra, the process for returns can be written as

\[ \frac{dS_t + D_t dt}{S_t} - r_t dt = \left( \mu_D - r(s_t) + \frac{\sigma_s(s_t)^2}{2} f''(s_t) \frac{f'(s_t)}{f(s_t)} + (\mu_s(s_t)
\text{+ } \sigma_D \sigma_s(s_t)) f'(s_t) + \frac{1}{f(s_t)} \right) dt + \left( \sigma_D + \frac{f'(s_t)}{f(s_t)} \sigma_s(s_t) \right) dB_t. \]

Comparison of this equation with equation (2) gives the statement of the lemma. 

Q.E.D.
The equilibrium market price of risk is rewritten as pricing equations relating the interest rate investors are unconstrained in the equilibrium. It therefore follows that the lemma of type A investors. Although the solution to this problem is described by Proposition 1, it is more convenient to explicitly use the fact that type A investors solve the following system of equations:

\[
\begin{align*}
\frac{1}{2} \sigma_s(s)^2 f''(s) + \left( \mu_s(s) + (1 - \gamma_A)\sigma_D\sigma_s(s) + \frac{\gamma_A \sigma_s(s)^2}{1 - s} \right) f'(s) \\
+ \left( \mu_D - r(s) - \gamma_A\sigma_D^2 + \frac{\gamma_A \sigma_D \sigma_s(s)}{1 - s} \right) f(s) + 1 &= 0, \\
\sigma_s(s)^2 f''(s) + \frac{1}{\gamma_B} f'(s)^2 + \frac{H(s)}{\gamma_B} \left[ \mu_s(s) + (1 - \gamma_B)\sigma_s(s) \left( \sigma_D + \frac{\sigma_s(s)}{s} \right) \right] - \beta \\
+ (1 - \gamma_B) \left[ \frac{\mu_s(s) + (1 - \gamma_B)\sigma_D\sigma_s(s)}{s} \right] + \mu_D - \frac{\gamma_B}{2} \left( \sigma_D^2 + \frac{\sigma_s(s)^2}{s^2} \right) \\
+ \exp \left( -\frac{H(s)}{\gamma_B} \right) &= 0, \\
\sigma_D + \frac{\sigma_s(s)}{s} &= \min \left[ \tilde{\omega}(\sigma_s(s), \sigma_Q(s), \ldots) \sigma_Q(s) - \sigma_s(s) \frac{H'(s)}{\gamma_B}, \frac{\gamma_A}{\gamma_B} \left( \sigma_D - \frac{\sigma_s(s)}{1 - s} \right) \right], \\
\mu_s(s) &= \left( \frac{\gamma_A}{1 - s} + \frac{\gamma_B}{s} \right)^{-1} \left[ (\gamma_A - \gamma_B) \left( \mu_D + \frac{\sigma_D \sigma_s(s)}{s} \right) \\
+ \frac{\gamma_A(1 - \gamma_A)}{2} \left( \sigma_D - \frac{\sigma_s(s)}{1 - s} \right)^2 - \frac{\gamma_B(1 - \gamma_B)}{2} \left( \sigma_D + \frac{\sigma_s(s)}{s} \right)^2 - \frac{\gamma_A \sigma_s(s)^2}{s(1 - s)^2} \right] \\
+ \left( (\gamma_A - \gamma_B)\sigma_D - \sigma_s(s) \left( \frac{\gamma_A}{1 - s} + \frac{\gamma_B}{s} \right) \right) \sigma_s(s) \frac{H'(s)}{\gamma_B}], \\
r(s) &= \beta + \gamma_A \mu_D - \frac{\gamma_A(\gamma_A + 1)}{2} \sigma_D^2 - \frac{\gamma_A}{1 - s} \left( \mu_s(s) - \gamma_A \sigma_D \sigma_s(s) \right) - \frac{\gamma_A(\gamma_A + 1)\sigma_s(s)^2}{2(1 - s)^2}. \\
\end{align*}
\]

The equilibrium market price of risk is

\[
\eta(s) = \gamma_A \left( \sigma_D - \frac{\sigma_s(s)}{1 - s} \right). 
\]

To derive these equations, consider first the consumption and portfolio problem of type A investors. Although the solution to this problem is described by Proposition 1, it is more convenient to explicitly use the fact that type A investors are unconstrained in the equilibrium. It therefore follows that the market for such investors is complete and their first-order conditions can be rewritten as pricing equations relating the interest rate \( r(s_t) \) and expected excess returns \( \mu_Q(s_t) \) to the characteristics of their consumption process \( C_i^A \).
For CRRA preferences, the discount factor is \( \Lambda_t = \exp(-\beta t) u'(C_t^A) \), the functions \( r(s) \) and \( \mu_Q(s) \) are given by the standard formulas:

\[
 r(s_t) = -\frac{1}{dt} E_t \left( \frac{d\Lambda_t}{\Lambda_t} \right), \quad \mu_Q(s_t) = -\frac{1}{dt} E_t \left( \frac{d\Lambda_t}{\Lambda_t} \frac{dS_t}{S_t} \right). \tag{A11}
\]

For CRRA preferences, the discount factor is \( \Lambda_t = \exp(-\beta t)(C_t^A)^{-\gamma_A} \) and, hence,

\[
 d\Lambda_t \quad \Lambda_t = -\beta dt - \gamma_A \frac{dC_t^A}{C_t^A} + \frac{\gamma_A(\gamma_A + 1)}{2} \left( \frac{dC_t^A}{C_t^A} \right)^2.
\]

In terms of the consumption share \( s_t \), the consumption of type A investors is \( C_t^A = (1 - s_t)D_t \), Ito's lemma applied to this equation yields

\[
 \frac{dC_t^A}{C_t^A} = \left( \mu_D - \frac{\mu_D(s_t) + \sigma_D \sigma_A(s_t)}{1 - s_t} \right) dt + \left( \sigma_D - \frac{\sigma_D(s_t)}{1 - s_t} \right) dB_t,
\]

and therefore

\[
 \frac{d\Lambda_t}{\Lambda_t} = -\beta dt - \gamma_A \left( \mu_D - \frac{\mu_D(s_t) + \sigma_D \sigma_A(s_t)}{1 - s_t} \right) dt + \left( \sigma_D - \frac{\sigma_D(s_t)}{1 - s_t} \right) dB_t.
\]

Using equation (A11), it is easy to find the interest rate \( r(s_t) \) and expected excess returns \( \mu_Q(s_t) \):

\[
 r(s_t) = \beta + \gamma_A \mu_D - \gamma_A(\gamma_A + 1) \sigma_D^2 - \frac{\gamma_A}{1 - s_t} \left( \sigma_D^2 - \gamma_A \sigma_A \sigma_D(s_t) - \frac{\gamma_A(\gamma_A + 1) \sigma_A(s_t)^2}{2(1 - s_t)^2} \right),
\]

\[
 \mu_Q(s_t) = \gamma_A \left( \sigma_D - \frac{\sigma_A(s_t)}{1 - s_t} \right) \sigma_Q(s_t). \tag{A12}
\]

Equation (A12) coincides with (A9). Equation (A13) effectively characterizes the market price of risk \( \eta(s_t) = \mu_Q(s_t)/\sigma_Q(s_t) \) and coincides with (A10).

The equation for the price–dividend ratio \( f(s) \) is derived by combining equations (10) and (11) with equation (A13). Next, the definition of \( s_t \) implies that \( C_t^B = s_t D_t \), and Ito's lemma yields

\[
 \frac{dC_t^B}{C_t^B} = \left( \frac{\mu_A(s_t)}{s_t} + \mu_D + \frac{\sigma_D \sigma_A(s_t)}{s_t} \right) dt + \left( \sigma_D + \frac{\sigma_A(s_t)}{s_t} \right) dB_t. \tag{A14}
\]

Equation (7) gives the optimal consumption–wealth ratio of type B investors. Denoting it by \( h(s) \), the wealth process of type B investors can be written as

\[
 \frac{dW_t^B}{W_t^B} = (r(s_t) - h(s_t)) dt + \omega^B(s_t)(\mu_Q(s_t) dt + \sigma_Q(s_t) dB_t). \tag{A15}
\]
Equation (A15) together with Ito's lemma applied to the consumption–wealth ratio yields
\[
\frac{dC^B_t}{C^B_t} = \left[ \frac{h''(s_t)}{h(s_t)} \sigma_s(s_t)^2 + \frac{h'(s_t)}{h(s_t)} (\mu_s(s_t) + \sigma_s(s_t)\omega^B(s_t)\sigma_Q(s_t)) \right] dt + \left[ \frac{h'(s_t)}{h(s_t)} \sigma_s(s_t) + \omega^B(s_t)\sigma_Q(s_t) \right] dB_t.
\]
(A16)

The optimality of consumption implies that the processes in equations (A14) and (A16) are identical. Hence, their drifts and diffusions should coincide, that is,
\[
\frac{\mu_s + \sigma_D\sigma_s}{s} = r - \mu_D - h + \omega^B\mu_Q + \frac{h''}{2h}\sigma_s^2 + \frac{h'}{h} (\mu_s + \sigma_s\omega^B\sigma_Q),
\]
(A17)
\[
\sigma_D + \frac{\sigma_s}{s} = \frac{h'}{h} \sigma_s + \omega^B\sigma_Q.
\]
(A18)
where the argument \( s_t \) has been omitted to simplify notation. Using the definition of \( h \), equations (A17) and (A18) can be rewritten as
\[
\omega^B\mu_Q = (\mu_s + \sigma_D\sigma_s) \left( \frac{1}{s} + \frac{H'}{\gamma B} \right) - r + \mu_D + \exp \left( -\frac{H}{\gamma B} \right) + \frac{\sigma_s^2}{2\gamma B} \left( H'' + \frac{1}{\gamma B} (H')^2 \right)
\]
\[+ \frac{H'\sigma_s^2}{\gamma Bs}, \]
(A19)
\[
\omega^B\sigma_Q = \sigma_D + \frac{\sigma_s}{s} + \frac{H'}{\gamma Bs} \sigma_s.
\]
(A20)
The substitution of equations (A19) and (A20) into (9) gives
\[
\frac{1}{2\gamma B} \left( \frac{H''}{\gamma B} + \frac{1}{\gamma B} (H')^2 \right) \sigma_s^2 + \mu_s \left( \frac{1}{s} + \frac{H'}{\gamma B} - \frac{\gamma B}{s} \right) + (1 - \gamma B)\sigma_s \left( \sigma_D + \frac{\sigma_s}{s} \right) \frac{H'}{\gamma B}
\]
\[= \frac{\gamma B(1 - \gamma B)}{2} \left( \sigma_D + \frac{\sigma_s}{s} \right)^2 + (1 - \gamma B) \left( \mu_D + \frac{\sigma_D\sigma_s}{s} \right) - \beta + \exp \left( -\frac{H(s)}{\gamma B} \right) = 0.
\]
(A21)

This is equation (A6). To get one more equation relating the functions \( H \) and \( \mu_s \), combine equations (A19) and (A20) as
\[
(\mu_s + \sigma_D\sigma_s) \left( \frac{1}{s} + \frac{H'}{\gamma B} \right) - r + \mu_D + \exp \left( -\frac{H}{\gamma B} \right) + \frac{\sigma_s^2}{2\gamma B} \left( H'' + \frac{1}{\gamma B} (H')^2 \right) + \frac{H'\sigma_s^2}{\gamma Bs}
\]
\[= \frac{\mu_Q}{\sigma_Q} \left( \sigma_D + \frac{\sigma_s}{s} + \frac{H'}{\gamma B} \sigma_s \right).
\]
Substituting the market price of risk from (A13) and the risk-free rate from (A12), and doing some algebra, it is easy to get another equation for \( H \):

\[
\frac{1}{2\gamma_B} \left( H'' + \frac{1}{\gamma_B} (H')^2 \right) \sigma_s^2 + \mu_s \left( \frac{1}{s + H' \gamma_A \sigma_s} + \frac{1 - \gamma_A \sigma_s}{\gamma_B} + (1 - \gamma_A) \sigma_s \left( \frac{\sigma_d}{s} + \frac{\sigma_f}{s} \right) \right) H' \\
- \frac{\gamma_A (1 - \gamma_A)}{2} \left( \sigma_d - \frac{\sigma_s}{1 - s} \right)^2 + (1 - \gamma_A) \left( \mu_d + \frac{\sigma_d \sigma_s}{s} \right) - \beta + \exp \left( -\frac{H(s)}{\gamma_B} \right) \\
+ \frac{\gamma_A \sigma_s^2}{s(1 - s)} \left( \frac{1}{1 - s} + \frac{H'}{\gamma_B} \right) = 0.
\]

(A22)

An explicit formula for \( \mu_s \) is obtained by subtracting equation (A22) from (A21):

\[
\mu_s = \left( \frac{\gamma_A}{1 - s} + \frac{\gamma_B}{s} \right)^{-1} \left[ (\gamma_A - \gamma_B) \sigma_s \left( \sigma_d + \frac{\sigma_s}{s} \right) \frac{H'}{\gamma_B} + (\gamma_A - \gamma_B) \left( \mu_d + \frac{\sigma_d \sigma_s}{s} \right) \\
+ \frac{\gamma_A (1 - \gamma_A)}{2} \left( \sigma_d - \frac{\sigma_s}{1 - s} \right)^2 - \gamma_B (1 - \gamma_B) \left( \sigma_d + \frac{\sigma_s}{s} \right)^2 \\
- \frac{\gamma_A \sigma_s^2}{s(1 - s)} \left( \frac{1}{1 - s} + \frac{H'}{\gamma_B} \right) \right].
\]

Collecting the terms with \( H' \) gives equation (A8). To obtain (A7), it is sufficient to combine equations (8) and (A20), taking into account the form of the margin constraints from (5) and the market price of risk from (A13).

Q.E.D.

**Proof of Proposition 4:** The equilibrium functions \( \Sigma_s(s, u), \mu_s(s, u), r(s, u), \) and \( \eta(s, u) \) have the following forms:

\[
\Sigma_s = \begin{cases} 
\left( \frac{1}{\gamma_A} + \frac{\gamma_B}{s} \right)^{-1} (\gamma_A - 1) \Sigma_D, & (s, u) \in \mathcal{A}^0, \\
\left( \frac{1}{\gamma_A} + \frac{\gamma_B}{s} \right)^{-1} (\gamma_A - 1) \Sigma_D - \frac{1}{2} \sigma_D s \left( \frac{\gamma_A}{1 - s + \gamma_A s} - \bar{m} \right) [1 \ 1], & (s, u) \in \mathcal{A}^1, \\
[\sigma_D s (\bar{m} - u) \quad \sigma_D s (u - 1)], & (s, u) \in \mathcal{A}^{12}, \\
[-\sigma_D s u \quad \sigma_D s (\bar{m} - 1 + u)], & (s, u) \in \mathcal{A}^{13}, 
\end{cases}
\]

(A23)

\( \mathcal{A}^0 = \{(s, u) : \gamma_A \leq \bar{m}(1 - s + \gamma_A s)\} \),

\( \mathcal{A}^1 = \{(s, u) : \gamma_A(2u - 1) < \bar{m}(1 - s + \gamma_A s), \ \gamma_A(1 - 2u) < \bar{m}(1 - s + \gamma_A s), \ \gamma_A > \bar{m}(1 - s + \gamma_A s)\} \),

\( \gamma_A > \bar{m}(1 - s + \gamma_A s) \),

\( \mathcal{A}^{12} = \{(s, u) : \gamma_A(2u - 1) > \bar{m}(1 - s + \gamma_A s), \ \gamma_A > \bar{m}(1 - s + \gamma_A s)\} \),

\( \mathcal{A}^{13} = \{(s, u) : \gamma_A(1 - 2u) > \bar{m}(1 - s + \gamma_A s), \ \gamma_A > \bar{m}(1 - s + \gamma_A s)\} \),
\[ \mu_s = \left( \frac{1}{s} + \frac{\gamma_A}{1-s} \right)^{-1} \left[ (\gamma_A - 1) \left( \mu_D + \frac{1}{s} \Sigma_s \Sigma'_D \right) - \frac{\gamma_A}{s(1-s)^2} \Sigma_s \Sigma'_s \right] - \frac{\gamma_A(\gamma_A - 1)}{2} \left( \Sigma_D - \frac{1}{1-s} \Sigma_s \right) \left( \Sigma_D - \frac{1}{1-s} \Sigma_s \right)' \].  

(A24)

\[ r = \beta + \gamma_A \mu_D - \frac{\gamma_A}{1-s} (\mu_s + \Sigma_s \Sigma'_D) - \frac{\gamma_A(\gamma_A + 1)}{2} \left( \Sigma_D - \frac{1}{1-s} \Sigma_s \right) \times \left( \Sigma_D - \frac{1}{1-s} \Sigma_s \right)' \].  

(A25)

\[ \eta = \gamma_A \left( \Sigma_D - \frac{1}{1-s} \Sigma_s \right)' \].  

(A26)

The partial differential equations for the functions \( f_i(s, u), i = 1, 2, \) are

\[ \frac{1}{2} f_{i ss} \Sigma_s \Sigma'_s + \frac{1}{2} f_{i uu} \Sigma_u \Sigma'_u + f_{i su} \Sigma_s \Sigma'_u + f_{i s} (\mu_s + \sigma_D \Sigma_u) + f_{i u} (\mu_u + \sigma_D \Sigma_u ) + (\mu_D - r - \mu_{Q_k}) f_i + 1 = 0, \quad i = 1, 2, \]  

(A27)

where the derivatives of the functions \( f_1 \) and \( f_2 \) are denoted by lower indices. The expected returns \( \mu_Q(s, u) \) and the diffusion part of returns \( \Sigma_Q(s, u) \) can be found from

\[ \Sigma_Q = \sigma_D I_2 + F_s \Sigma_s + F_u \Sigma_u, \]  

(A28)

\[ \mu_Q = \gamma_A \Sigma_Q \left( \Sigma_D - \frac{1}{1-s} \Sigma_s \right)' \]  

(A29)

where \( I_2 \) is a 2 x 2 identity matrix, \( F_s = [f_{1s}/f_1 \ f_{2s}/f_2]' \), and \( F_u = [f_{1u}/f_1 \ f_{2u}/f_2]' \). The optimal portfolio weights of type B investors in the equilibrium are

\[ \omega^B = \begin{cases} \frac{\gamma_A}{\gamma_A + \gamma_A s} \left( \Sigma'_Q \right)^{-1} \Sigma_D, & (s, u) \in A_0, \\ \left( \Sigma'_Q \right)^{-1} \left( \frac{\gamma_A}{\gamma_A + \gamma_A s} \left( \Sigma_D - \frac{1}{2} \Sigma_D [1 \ 1] \right) + \frac{1}{2} \tilde{\sigma} \Sigma_D [1 \ 1] \right)' , & (s, u) \in A_1, \\ \tilde{\sigma} \left( \Sigma'_Q \right)^{-1} [1 \ 0]' , & (s, u) \in A_{12}, \\ \tilde{\sigma} \left( \Sigma'_Q \right)^{-1} [0 \ 1]' , & (s, u) \in A_{13}, \end{cases} \]  

(A30)

where the regions \( A_0, A_1, A_{12}, \) and \( A_{13} \) are defined above. The optimal portfolio weights of type A investors are

\[ \omega^A = \frac{1}{\gamma_A} \left( \Sigma_Q \Sigma'_Q \right)^{-1} \left( \mu_Q + \Sigma_Q \Sigma_u H_u + \Sigma_Q \Sigma_s H_s \right), \]  

(A31)
where the function \( H(s, u) \) solves the following partial differential equation:

\[
\begin{align*}
\frac{1}{2} \left( H_{ss} + H_{s}^2 \right) \Sigma_s \Sigma_s' + \frac{1}{2} \left( H_{uu} + H_u^2 \right) \Sigma_u \Sigma_u' + (H_{us} + H_u H_s) \Sigma_u \Sigma_s' + H_s \mu_s + H_u \mu_u \\
+ \frac{1 - \gamma_A}{2 \gamma_A} \left( \mu_Q + \Sigma Q \Sigma_u' H_u + \Sigma Q \Sigma_s' H_s \right)' \left( \Sigma Q \Sigma_Q \right)^{-1} \left( \mu_Q + \Sigma Q \Sigma_u' H_u + \Sigma Q \Sigma_s' H_s \right) \\
+ \gamma_A \exp \left( -\frac{H}{\gamma_A} \right) + (1 - \gamma_A) r - \beta = 0.
\end{align*}
\]

(A32)

Several steps of the proof of Proposition 4 are similar to those of Propositions 1 and 2 and Lemma 1, and here many details are omitted.

First, consider the optimal consumption and portfolio problem of type A investors, who are assumed to be unconstrained. By the definition of the state variable \( s \), the consumption of type A investors is \( C_A = (1 - s) D \). Ito’s lemma applied to this equation together with equations (19) and (20) yields

\[
dC_A \over C_A = \left( \mu_D - \frac{\mu_s + \Sigma_s \Sigma_D'}{1 - s} \right) dt + \left( \Sigma_D - \frac{1}{1 - s} \Sigma_s \right) dB.
\]

For CRRA preferences, the discount factor is \( \Lambda = \exp(-\beta t)(C_A)^{-\gamma_A} \). Applying Ito’s lemma again and using the previous equation, it is straightforward to get

\[
\begin{align*}
\frac{d\Lambda}{\Lambda} &= -\beta dt - \gamma_A \left( \mu_D - \frac{\mu_s + \Sigma_s \Sigma_D'}{1 - s} - \frac{\gamma_A + 1}{2} \left( \Sigma_D - \frac{1}{1 - s} \Sigma_s \right) \right) \\
&\quad \times \left( \Sigma_D - \frac{1}{1 - s} \Sigma_s \right)' dt - \gamma_A \left( \Sigma_D - \frac{1}{1 - s} \Sigma_s \right) dB.
\end{align*}
\]

Equations (A25) and (A26) follow immediately. Since \( \mu_Q = \Sigma Q \eta \), equation (A26) yields (A29).

Next, consider the portfolio problem of type B (logarithmic) investors, who can be constrained in some states of the economy. At each moment, the set of attainable portfolio weights is \( \Omega = \{ \omega : \omega' m^K \leq 1, K = 1, \ldots, 4 \} \), where the investor type superscript for the portfolio weights is omitted without causing confusion. The value function satisfies the following Bellman equation:

\[
\max_{\{C^B, \omega \in \Omega\}} \left[ e^{-\beta t} \log(C^B) + DJ \right] = 0.
\]

(A33)

where \( DJ \) is given by

\[
DJ = J_W (r W - C^B + W \omega' \mu_Q) + J_t + \frac{1}{2} J_W W^2 \omega' \Sigma_Q \Sigma_Q' \omega' + \text{other terms from Ito’s lemma}
\]
and \( W \) is the wealth of type B investors. The value function is assumed to have a standard form:

\[
J(W, s, u, t) = \left( \frac{1}{\beta} \log W + H(s, u) \right) \exp(-\beta t). \tag{A34}
\]

Because the function \( H(s, u) \) shifts the level of the utility function but leaves the marginal utility intact, its form is not needed for characterization of the equilibrium. The maximization over \( C_B^B \) immediately yields \( C_B^B = \beta W \). Hence, 
\[
\frac{dC_B^B}{C_B^B} = \frac{dW}{W}.
\]

On the other hand, using \( C_B^B = sD \) and applying Ito’s lemma, it is easy to show that

\[
\frac{dC_B^B}{C_B^B} = \left( \mu_D + \frac{\mu_s + \Sigma_s \Sigma'_D}{s} \right) dt + \left( \Sigma_D + \frac{1}{s} \Sigma_s \right) dB. \tag{A35}
\]

The equality of drifts and diffusions in equations (A35) and (A36) yields

\[
\mu_D + \frac{\mu_s + \Sigma_s \Sigma'_D}{s} = r - \beta + \omega' \mu_Q, \tag{A37}
\]

\[
\omega' \Sigma_Q = \Sigma_D + \frac{1}{s} \Sigma_s. \tag{A38}
\]

First, consider equation (A37). Noting that \( \mu_Q = \Sigma_Q \eta \) and using (A26), we get

\[
\mu_D + \frac{\mu_s + \Sigma_s \Sigma'_D}{s} = r - \beta + \gamma_A \omega' \Sigma_Q \left( \Sigma_D - \frac{1}{1-s} \Sigma_s \right)',
\]

which together with equation (A38) yields

\[
\mu_D + \frac{\mu_s + \Sigma_s \Sigma'_D}{s} = r - \beta + \gamma_A \left( \Sigma_D + \frac{1}{s} \Sigma_s \right) \left( \Sigma_D - \frac{1}{1-s} \Sigma_s \right)'.
\]

Equation (A24) results from substituting \( r \) from equation (A25) and doing some algebra.

Next, the maximization over \( \omega \) in equation (A33) reduces to

\[
\max_{\omega \in \Omega_1} \left[ \omega' \mu_Q - \frac{1}{2} \omega' \Sigma_Q \Sigma'_Q \omega \right] = 0,
\]

which, using equation (A26), can be rewritten as

\[
\max_{\omega \in \Omega_1} \left[ \gamma_A \omega' \Sigma_Q \left( \Sigma_D - \frac{1}{1-s} \Sigma_s \right)' - \frac{1}{2} \omega' \Sigma_Q \Sigma'_Q \omega \right] = 0.
\]
Plugging the optimal portfolio weights into equation (A38), it is easy to obtain an equation for $\Sigma_s$:

$$
\Sigma_D' + \frac{1}{s} \Sigma_s' = \Sigma_q' \arg \max_{\omega \in \Omega} \left[ \gamma_A \omega' \Sigma Q \left( \Sigma_D - \frac{1}{1-s} \Sigma_s \right)' - \frac{1}{2} \omega' \Sigma Q \Sigma_q' \omega \right].
$$

Due to the choice of the constraints in the form $\Omega = \{\omega : \omega' \Sigma Q A^K \leq 1, K = 1, \ldots, 4\}$, where $A^K, K = 1, \ldots, 4$, are constant matrices, equation (A39) has a closed-form solution. To find it, sequentially consider several possible cases distinguished by the sets of constraints that bind in the optimization problem in equation (A39). Note that no more than two constraints can bind simultaneously, so only the following combinations of constraints should be considered: $\emptyset$, $\{K, K = 1, \ldots, 4\}$, and $\{KL, K = 1, \ldots, 4; L = 1, \ldots, 4; K < L\}$.

**Case 1: $\emptyset$.** In this case, the optimum in equation (A39) is an internal point, so

$$
\omega = \gamma_A (\Sigma_Q')^{-1} \left( \Sigma_D - \frac{1}{1-s} \Sigma_s \right)'
$$

and equation (A39) gives

$$
\Sigma_D + \frac{1}{s} \Sigma_s = \gamma_A \left( \Sigma_D - \frac{1}{1-s} \Sigma_s \right).
$$

The solution for $\Sigma_s$ is

$$
\Sigma_s = \left( \frac{1}{s} + \frac{\gamma_A}{1-s} \right)^{-1} (\gamma_A - 1) \Sigma_D
$$

and the optimal portfolio weights are

$$
\omega = \frac{\gamma_A}{1-s + \gamma_A s} (\Sigma_Q')^{-1} \Sigma_D'.
$$

This is the first line in equation (A30). The unconstrained case obtains when these weights satisfy all constraints, that is,

$$
\frac{\gamma_A}{1-s + \gamma_A s} \Sigma_D A^K \leq 1, \quad K = 1, \ldots, 4.
$$

The definitions of $\Sigma_D$ and $A^K, K = 1, \ldots, 4$, imply that

$$
\Sigma_D A^1 = \frac{1}{\bar{m}}, \quad \Sigma_D A^2 = \frac{2u - 1}{\bar{m}}, \quad \Sigma_D A^3 = \frac{1 - 2u}{\bar{m}}, \quad \text{and} \quad \Sigma_D A^4 = -\frac{1}{\bar{m}}.
$$

Because $u \in (0, 1)$ and $\gamma_A/(1-s + \gamma_A s) > 0$, the first constraint in equation (A41) is the most restrictive and, if it is satisfied, all other constraints are
The solution with no binding constraints exists in the region \((s, u) \in A^0\),

\[ A^0 = \{(s, u) : \gamma_A \leq \bar{m}(1 - s + \gamma_A s)\}, \]

and the solution to equation (A39) is given by (A40).

**Case 2:** \(\{K, K = 1, \ldots, 4\}\). In this case, only the constraint \(K\) binds in equation (A39) and \(s < (\gamma_A - \bar{m})/(\bar{m}(\gamma_A - 1))\). The optimal portfolio weights maximize the Lagrange function

\[ \mathcal{L} = \gamma_A \omega' \Sigma_Q \left( \Sigma_D - \frac{1}{1 - s} \Sigma_s \right)' - \frac{1}{2} \omega' \Sigma_Q \Sigma_Q' \omega + \lambda(1 - \omega' \Sigma_Q A^K), \]

where \(\lambda > 0\) is a Lagrange multiplier associated with the constraint \(K\). The first-order conditions yield

\[ \omega = (\Sigma_Q')^{-1} \left( \gamma_A \left( \Sigma_D - \frac{1}{1 - s} \Sigma_s \right)' - \lambda A^K \right). \]  
(A43)

These weights should satisfy \(\omega' \Sigma_Q A^K = 1\), so, using the definition of matrices \(A^K\),

\[ \lambda = \frac{1}{2} \bar{m}^2 \sigma_D^2 \left( \gamma_A \left( \Sigma_D - \frac{1}{1 - s} \Sigma_s \right) A^K - 1 \right). \]  
(A44)

Plugging the optimal portfolio weights in equation (A39) gives an equation for \(\Sigma_s\):

\[ \Sigma_D + \frac{1}{s} \Sigma_s = \gamma_A \left( \Sigma_D - \frac{1}{1 - s} \Sigma_s \right) - \frac{1}{2} \bar{m}^2 \sigma_D^2 \left( \gamma_A \left( \Sigma_D - \frac{1}{1 - s} \Sigma_s \right) A^K - 1 \right) \Sigma_s. \]

Multiplying this equation by \(A^K\) yields \(\Sigma_s A^K = s(1 - \Sigma_D A^K)\), so the solution \(\Sigma_s\) is

\[ \Sigma_s = \left( \frac{1}{s} + \frac{\gamma_A}{1 - s} \right)^{-1} (\gamma_A - 1) \Sigma_D - \frac{1}{2} \bar{m}^2 \sigma_D^2 s \left( \frac{\gamma_A}{1 - s + \gamma_A s} \Sigma_D A^K - 1 \right) A^K. \]
(A45)

Next, it is necessary to check that \(\lambda > 0\) and that the other constraints except the constraint \(K\) are satisfied as inequalities. Using equations (A44) and (A45), the constraint \(\lambda > 0\) reduces to

\[ \Sigma_D A^K > s + \frac{1 - s}{\gamma_A}. \]  
(A46)

Comparison of equations (A42) and (A46) shows that equation (A46) is never satisfied for \(K = 4\), always satisfied for \(K = 1\) in the considered interval for \(s\), and can be satisfied for \(K = 2\) and \(K = 3\) when \(\gamma_A(2u - 1) > \bar{m}(1 - s + s\gamma_A)\) and \(\gamma_A(1 - 2u) > \bar{m}(1 - s + s\gamma_A)\), respectively.

It is a bit more cumbersome to check that the optimal portfolios satisfy the other constraints, that is, \(\omega' \Sigma_Q A^L < 1\) for \(L = 1, \ldots, 4, L \neq K\). Using equation...
Due to the violation of \( \lambda \) use (A43), these constraints can be written as

\[
\left( \gamma_A \left( \Sigma_D - \frac{1}{1-s} \Sigma_s \right) - \lambda A^{K'} \right) A^L < 1. \quad (A47)
\]

Noting that \( \Sigma_s \) can be represented as

\[
\Sigma_s = \left( \frac{1}{s} + \frac{\gamma_A}{1-s} \right)^{-1} \left( (\gamma_A - 1) \Sigma_D - \lambda A^{K'} \right), \quad (A48)
\]

it is possible to rewrite the inequalities in (A47) as

\[
(\gamma_A \Sigma_D - \lambda (1-s) A^{K'}) A^L < 1 - s + \gamma_A s. \quad (A49)
\]

Sequentially consider the cases \( K = 1, K = 2, \) and \( K = 3 \) \( (K = 4 \) is excluded due to the violation of \( \lambda > 0 \)).

\( K = 1 \): when \( L = 2 \), equation (A49) reduces to \( \gamma_A (2u - 1) < \bar{m}(1 - s + \gamma_A s) \); when \( L = 3 \), equation (A49) reduces to \( \gamma_A (2u - 2u) < \bar{m}(1 - s + \gamma_A s) \); and when \( L = 4 \), equation (A49) is always satisfied (to demonstrate this, it is necessary to use \( \lambda = \bar{m}^2 \sigma_D^2 (\gamma_A / \bar{m} - (1 - s + \gamma_A s)) / (2(1 - s)) \)). Thus, in the region \( \gamma_A (2u - 1) < \bar{m}(1 - s + \gamma_A s) \) and \( \gamma_A (2u - 2u) < \bar{m}(1 - s + \gamma_A s) \), there is a solution

\[
\Sigma_s = \left( \frac{1}{s} + \frac{\gamma_A}{1-s} \right)^{-1} (\gamma_A - 1) \Sigma_D - \frac{1}{2} \sigma_D s \left( \frac{\gamma_A}{1-s + \gamma_A s} - \bar{m} \right) [1 \ 1]. \quad (A50)
\]

\( K = 2 \): when \( L = 1 \), equation (A49) reduces to \( \gamma_A < \bar{m}(1 - s + \gamma_A s) \), and this inequality is violated in the considered interval for \( s \). Hence, the case \( K = 2 \) cannot occur.

\( K = 3 \): when \( L = 1 \), equation (A49) reduces to \( \gamma_A < \bar{m}(1 - s + \gamma_A s) \), and this inequality is violated in the considered interval for \( s \). Hence, the case \( K = 3 \) cannot occur.

To summarize, the solution with one binding constraint exists in the region \((s, u) \in A^1\),

\[
A^1 = \{(s, u) : \gamma_A (2u - 1) < \bar{m}(1 - s + \gamma_A s), \ \gamma_A (1 - 2u) < \bar{m}(1 - s + \gamma_A s), \ \gamma_A > \bar{m}(1 - s + \gamma_A s)\},
\]

where the first constraint binds and \( \Sigma_s \) is given by equation (A50). The substitution of \( \Sigma_s \) in equations (A43) and (A44) with \( K = 1 \) yields the optimal portfolio stated in (A30).

**Case 3:** \( KL, \ K = 1, \ldots, 4; \ L = 1, \ldots, 4; \ K < L \). Although, in general, there may exist six pairs of simultaneously binding constraints, only four of them \((12, 13, 24, \text{and} \ 34)\) can actually occur because the constraints \( K = 1 \) and \( K = 2 \) are incompatible with the constraints \( L = 4 \) and \( L = 3 \), respectively. As in Case 2, only the states satisfying \( \gamma_A > \bar{m}(1 - s + \gamma_A s) \) are considered. The optimal
The portfolio weights maximize the Lagrange function

\[ L = \gamma_A \omega' \Sigma_Q \left( \Sigma_D - \frac{1}{1-s} \Sigma_s \right)' - \frac{1}{2} \omega' \Sigma_Q \Sigma_Q' \omega + \lambda^K (1 - \omega' \Sigma_Q A^K) + \lambda^L (1 - \omega' \Sigma_Q A^L), \]

where \( \lambda^K > 0 \) and \( \lambda^L > 0 \) are Lagrange multipliers associated with the constraints \( K \) and \( L \). The first-order conditions yield

\[ \omega = (\Sigma_Q')^{-1} \left( \gamma_A \left( \Sigma_D - \frac{1}{1-s} \Sigma_s \right)' - \lambda^K A^K - \lambda^L A^L \right). \]  

(A51)

These weights should satisfy \( \omega' \Sigma_Q A^K = 1 \) and \( \omega' \Sigma_Q A^L = 1 \). The latter conditions together with the definition of matrices \( A^K \) and \( A^L \) give a system of equations for \( \lambda^K \) and \( \lambda^L \):

\[ \frac{2}{\bar{m}^2 \sigma_D^2} \lambda^K - (A^L' A^K) \lambda^L = \gamma_A \left( \Sigma_D - \frac{1}{1-s} \Sigma_s \right) A^K - 1, \]
\[ \frac{2}{\bar{m}^2 \sigma_D^2} \lambda^L - (A^L' A^K) \lambda^K = \gamma_A \left( \Sigma_D - \frac{1}{1-s} \Sigma_s \right) A^L - 1. \]

Note that for all admissible combinations of constraints \( A^L' A^K = 0 \), so

\[ \lambda^K = \frac{\bar{m}^2 \sigma_D^2}{2} \left( \gamma_A \left( \Sigma_D - \frac{1}{1-s} \Sigma_s \right) A^K - 1 \right), \]
\[ \lambda^L = \frac{\bar{m}^2 \sigma_D^2}{2} \left( \gamma_A \left( \Sigma_D - \frac{1}{1-s} \Sigma_s \right) A^L - 1 \right). \]

The equations \( \omega' \Sigma_Q A^K = 1 \) and \( \omega' \Sigma_Q A^L = 1 \) together with equation (A39) imply that

\[ \left( \Sigma_D + \frac{1}{s} \Sigma_s \right) A^K = 1, \quad \left( \Sigma_D + \frac{1}{s} \Sigma_s \right) A^L = 1, \]  

(A52)

and therefore

\[ \Sigma_s A^K = s(1 - \Sigma_D A^K), \quad \Sigma_s A^L = s(1 - \Sigma_D A^L). \]  

(A53)

Hence, the inequalities \( \lambda^K > 0 \) and \( \lambda^L > 0 \) reduce to

\[ \gamma_A \Sigma_D A^K > 1 - s + \gamma_A s, \quad \gamma_A \Sigma_D A^L > 1 - s + \gamma_A s. \]

Using equation (A42), it is easy to see that both inequalities can be satisfied only for the pairs of constraints 12 and 13. For the first pair, this happens when \( \gamma_A > \bar{m}(1 - s + \gamma_A s) \) and \( \gamma_A (2u - 1) > \bar{m}(1 - s + \gamma_A s) \), and for the second pair, this happens when \( \gamma_A > \bar{m}(1 - s + \gamma_A s) \) and \( \gamma_A (1 - 2u) > \bar{m}(1 - s + \gamma_A s) \). Note that the first inequality in each case is satisfied automatically in the
considered range of \( s \). Equation (A53) provides explicit formulas for \( \Sigma_s \):

\[
\Sigma_s = [\sigma_D s (\bar{m} - u) \quad \sigma_D s (u - 1)],
\]  

(A54)

for the pair of binding constraints 12 and

\[
\Sigma_s = [-\sigma_D s u \quad \sigma_D s (\bar{m} - 1 + u)],
\]  

(A55)

for the pair of binding constraints 13. It is easy to check that the portfolio weights simultaneously satisfying the constraints \( \omega' \Sigma_Q A^1 = 1 \) and \( \omega' \Sigma_Q A^2 = 1 \) also satisfy the constraints \( \omega' \Sigma_Q A^3 < 1 \) and \( \omega' \Sigma_Q A^4 < 1 \), and the portfolio weights simultaneously satisfying the constraints \( \omega' \Sigma_Q A^1 = 1 \) and \( \omega' \Sigma_Q A^3 = 1 \) also satisfy the constraints \( \omega' \Sigma_Q A^2 < 1 \) and \( \omega' \Sigma_Q A^4 < 1 \).

To summarize, a solution with two simultaneously binding constraints exists in the region \((s, u) \in A_{12} \),

\[
A_{12} = \{(s, u) : \gamma_A (2u - 1) > \bar{m} (1 - s + \gamma_A s), \quad \gamma_A > \bar{m} (1 - s + \gamma_A s)\},
\]

where the first and second constraints bind and \( \Sigma_s \) is given by (A54), and in the region \((s, u) \in A_{13} \),

\[
A_{13} = \{(s, u) : \gamma_A (1 - 2u) > \bar{m} (1 - s + \gamma_A s), \quad \gamma_A > \bar{m} (1 - s + \gamma_A s)\},
\]

where the first and third constraints bind and \( \Sigma_s \) is given by (A55). The solutions to the systems of linear equations \( \omega' \Sigma_Q A^1 = 1 \), \( \omega' \Sigma_Q A^2 = 1 \) and \( \omega' \Sigma_Q A^3 = 1 \) for \( \omega \) give the optimal portfolios in \( A_{12} \) and \( A_{13} \) stated in (A30).

To derive equation (A27), represent the price of the risky asset \( i, i = 1, 2 \), as

\[
S_{it} = D_i f_i(s, u)
\]

and apply Ito’s lemma:

\[
\frac{dS_{it}}{S_{it}} = \frac{dD_{it}}{D_{it}} + \frac{df_i}{f_i} + \frac{dD_{it} df_i}{D_{it} f_i},
\]

where

\[
\frac{df_i}{f_i} = f_{is}(\mu_s dt + \Sigma_s dB_t) + f_{iu}(\mu_u dt + \Sigma_u dB_t) + \frac{1}{2} f_{iss} \Sigma_s' dt + \frac{1}{2} f_{ius} \Sigma_u' dt + f_{iis} \Sigma_s' dt + f_{iui} \Sigma_u' dt + f_{ius} \Sigma_u \Sigma_s' dt
\]

Using the form of the dividend process from equation (17) and performing simple algebra, the process for returns can be written as

\[
\frac{dS_{it} + D_{it} dt}{S_{it}} - r_t dt = \left( \mu_D - r + \frac{1}{2} f_{iss} \Sigma_s' + \frac{1}{2} f_{iui} \Sigma_u' + f_{iis} \Sigma_s' + f_{iui} \Sigma_u \Sigma_s' \right)
\]

\[
+ \left( \mu_s + \sigma_D \Sigma_s dW_t + \mu_u + \sigma_D \Sigma_u dW_t \right) dt
\]

\[
+ \left( \sigma_D e_i + f_{iis} \Sigma_s + f_{iui} \Sigma_u \right) dB_t,
\]
where \( e_1 = [1 \ 0] \) and \( e_2 = [0 \ 1] \). Comparison of the diffusions in this equation and in equation (18) yields

\[
\Sigma Q_k = \sigma D e_1 + F_{si} \Sigma s + F_{ui} \Sigma u,
\]

where \( F_s = [f_{1s}/f_1 \ f_{2s}/f_2]' \) and \( F_u = [f_{1u}/f_1 \ f_{2u}/f_2]' \). This equation coincides with (A28). Similarly, comparison of the drifts results in

\[
\mu Q_k = \mu D - r + \frac{1}{2} \frac{f_{iss}}{f_i} \Sigma s' \Sigma s + \frac{1}{2} \frac{f_{iuu}}{f_i} \Sigma u' \Sigma u + \frac{1}{2} \frac{f_{ius}}{f_i} \Sigma s' \Sigma u + (\mu_u + \sigma D \Sigma u) \frac{f_{is}}{f_i},
\]

and this equation can be rewritten as equation (A27).

Finally, consider the portfolio problem of type A investors, who are assumed to be unconstrained. Their value function satisfies the following Bellman equation:

\[
\max_{[C, \omega]} \left[ e^{-\beta t} \frac{1}{1-\gamma_A} C^{1-\gamma_A} + DJ \right] = 0, \tag{A56}
\]

where \( DJ \) is given by

\[
DJ = J_W(W - C + W \omega' \mu_Q) + \frac{1}{2} J_{WW} W^2 \omega' \Sigma Q \Sigma Q' \omega + \frac{1}{2} J_{ss} \Sigma s \Sigma s' + \frac{1}{2} J_{uu} \Sigma u \Sigma u' + J_{ws} \omega' \Sigma s + J_{us} \Sigma s \omega + J_{uu} \mu u + J_t.
\]

To simplify notation, the subscripts indicating the investor’s type have been omitted. Given that the investors have CRRA preferences, the value function is assumed to have the form

\[
J(W, s, u, t) = \frac{1}{1-\gamma_A} W^{1-\gamma_A} \exp \left( H(s, u) \right) \exp(-\beta t). \tag{A57}
\]

Maximization in the Bellman equation over \( C \) immediately yields \( C = W \exp(-H(s, u)/\gamma_A) \). Unconstrained maximization over \( \omega \) together with the assumed form of the value function gives (A31). Equation (A32) results from substituting the optimal consumption and the optimal portfolio weights in (A56) and doing algebraic manipulations. This completes the proof. Q.E.D.

**Appendix B: Numerical Techniques**

This appendix reports details on numerical techniques that I use to solve the nonlinear differential equations in Sections II and III. The key idea is to apply the projection method, which is essentially equivalent to looking for an approximate solution in the form of a linear combination of orthogonal polynomials (Judd (1998)). As an orthogonal basis, I use Chebyshev polynomials of the first kind.
Consider first the case in which the unknown function has one argument (in Section II it is the state variable \( s \)). Because Chebyshev polynomials form an orthogonal basis in \( L^2([-1, 1]) \), the state variable \( s \) defined on the interval \([0, 1]\) is rescaled as
\[
z = 2s - 1,
\]
so the range of \( z \) is \([-1, 1]\). Clearly, this transformation monotonically and smoothly maps the interval \([0, 1]\) into the interval \([-1, 1]\): the boundary \( s = 0 \) corresponds to \( z = -1 \) and \( s = 1 \) corresponds to \( z = 1 \). The change of variables also affects the derivatives appearing in the differential equations:
\[
\frac{\partial}{\partial z} = \frac{1}{2} \frac{\partial}{\partial s}, \quad \frac{\partial^2}{\partial z^2} = \frac{1}{4} \frac{\partial^2}{\partial s^2}.
\]

Denote the solution to the given system of differential equations as \( \{Y^i(z)\} \) (in Section II. I apply the projection method to equations (A5) and (A6), so \( \{Y^1(z), Y^2(z)\} \equiv \{f(z), H(z)\} \)). The projection method approximates the solution by a linear combination of the Chebyshev polynomials \( \{T_j(z)\}, j = 0, \ldots, N \), that is,
\[
Y^i(z) = \sum_{j=0}^{N} a^i_j T_j(z),
\]
where \( a^i_j \) are unknown constants. The highest order of the Chebyshev polynomial \( N \) used in the decomposition of \( Y^i(z) \) determines the quality of the approximation. The coefficients \( a^i_j \) should be chosen so that \( Y^i(z) \) minimizes the deviation of the left-hand side of the differential equations from zero, so an infinite-dimensional functional search is reduced to a finite-dimensional minimization problem.

To set the objective function, I use the overidentified collocation method, which minimizes the sum of squared errors computed at the points \( \{z_m, m = 0, \ldots, M\} \), where \( M > N \). For a better approximation, I choose \( \{z_m, m = 0, \ldots, M\} \) to be a Chebyshev array, that is, a set of points where \( T_M(z) = 0 \). Note that the optimization problem is overidentified, so when the value of the objective function in the minimization problem is close to zero in the optimum, the solution to the differential equations is likely to exist and the projection method provides a reasonable approximation of it. In the practical implementation of the projection method, I set \( M = 50 \) and \( N = 30 \). For these parameters, the approximation error has order \( 10^{-7} \). Moreover, the results are stable with respect to the variation in the highest degree of Chebyshev polynomials \( N \) and in the number of collocation points \( M \). This is additional evidence that the numerical approximation indeed converges to the exact solution.

In Section III, I apply the projection method to solve equations (A27) and (A32), which are partial differential equations. Their solutions \( f_1, f_2, \) and \( H \) are functions of the two state variables \( s \) and \( u \), which are defined on \([0, 1]\) and
mapped on the interval $[-1, 1]$ by $z_1 = 2s - 1$ and $z_2 = 2u - 1$. The solutions to the differential equations are approximated by

$$Y^i(z_1, z_2) = \sum_{j=0}^{N_1} \sum_{k=0}^{N_2} a_{jk}^i T_j(z_1) T_k(z_2),$$

where $a_{jk}^i$ are the constants to find. As in the one-dimensional case, I use the collocation method with the Chebyshev array and minimize the sum of squared errors produced by the left-hand side of the differential equations at the points $\{z_1 = z_l, z_2 = z_m, \ l, m = 0, \ldots, M\}$, where $M = 50$. To maintain numerical tractability, the highest polynomial orders are chosen to be $N_1 = 15$ and $N_2 = 8$. This choice produces approximation error of order $10^{-5}$.

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**Appendix S1:** Internet Appendix